SETS, MODELS, AND PROOFS: TOPICS IN THE THEORY OF RECURSIVE FUNCTIONS

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We prove results in various areas of recursion theory. First, in joint work with Richard Shore, we prove a new jump-inversion result for ideals of recursively enumerable (r.e.) degrees; this defeats what had seemed to be a promising tack on the automorphism problem for the semilattice $\mathcal{R}$ of r.e. degrees.

Second, in work spanning two chapters, we calibrate the reverse-mathematical strength of a number of theorems of basic model theory, such as the Ryll-Nardzewski atomic-model theorem, Vaught's no-two-model theorem, Ehrenfeucht's three-model theorem, and the existence theorems for homogeneous and saturated models. Whereas most of these are equivalent over RCA_0 to one of RCA_0, WKL_0, ACA_0, as usual, we also uncover model-theoretic statements with exotic complexities such as $\neg\mathsf{WKL}_0 \vee \mathsf{ACA}_0$ and $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$.

Third, we examine the possible weak truth table (wtt) degree spectra of countable first-order structures. We find several points at which the wtt- and Turing-degree cases differ, notably that the most direct wtt analogue of Knight's dichotomy theorem does not hold. Yet we find weaker analogies between the two, including a new *trichotomy* theorem for wtt degree spectra in the spirit of Knight's.

BIOGRAPHICAL SKETCH

David Belanger was born and raised in Belleville, Ontario, Canada. He attended the University of Waterloo, where he earned Bachelor's and Master's degrees in mathematics. During his Master's studies he concentrated on the area of mathematical logic, and in 2009 he moved to Cornell University for a Ph.D. in the same.

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Last and most important are family, near and far, but never estranged; spanning provinces, countries, and now continents; among whom I am most at home, and without whom I could never get very far.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 0

GLOBAL INTRODUCTION

In the course of developing a mathematical theory, it is natural to consider the complexity of the examples and counterexamples that are supposed to be produced. The precise definition of simplicity and complexity may vary. Some well-known notions of a simple object are: an equation that can be solved by radicals; a length that can be constructed using a compass and straightedge; a subset of $\mathbb{R}$ that is Borel; an optimization problem that can be solved by a computer program in polynomial time; and a subset of $\mathbb{N}$ that can be computed by some effective procedure. These are inequivalent in general, and each is a reasonable definition in its proper context. *Recursion theory* is the area of mathematical logic that studies the last of these—the effective computability of sets—and related questions of effective enumerability and relative computability.

Following the publication of Turing's 1936 article [70], the community has generally come to agree that an effective procedure is one which is computable by one of his abstract *Turing machines.* The *idea* of an effective procedure, however, far predates any such consensus. In 1900, for instance, Hilbert asked as the tenth of his famous Problems for a procedure to determine whether a given multivariate polynomial has integer roots. Such a procedure, if found, would constitute a positive solution with no need for a formal definition. The problem was eventually solved in the *negative*, however, when it was shown through the combined efforts of Matiyasevich, Robinson, Davis, and Putnam that no suitable Turing machine exists.

With Turing's definition as a starting point, mathematicians have made rapid progress on relatively concrete decision problems—obtaining negative solutions along the lines of Hilbert's tenth problem to the word problem for groups, to the problem of logical entailment, and to many more besides—but also on computability in the abstract, such as the theorem of Kleene and Post that there exists a pair of subsets of $\mathbb{N}$ neither of which can be used to compute the other. Recursion theory today focuses more on the latter type of question, but the strength of this preference varies among the numerous subfields. In what follows we will consider the semilattice of recursively enumerable degrees (a topic of pure recursion theory), recursive model theory (where recursion theory meets classical model theory), and reverse mathematics (where recursion theory meets proof theory, and where classical decision problems frequently reappear).

This dissertation collects all of the mathematical research that the author submitted for publication during the course of his doctoral studies. Not counting the present introductory Chapter 0, there are four chapters: each comprising, more or less unchanged, one journal article. The corresponding citations are provided at the beginning of each chapter. With the exception of Chapter 1, which was written in collaboration with Richard Shore, all of what follows is my work alone.

0.1 Recursive sets and Turing degrees

A set $A \subseteq \mathbb{N}$ of natural numbers is *recursive* if there is a Turing machine which, given input $n \in \mathbb{N}$, outputs either 1, if $n \in A$, or 0, if $n \notin A$. A set $A \subseteq \mathbb{N}$ is *recursively enumerable (r.e.)* if there is a Turing machine which on input n outputs 1 if $n \in A$, and fails to terminate (i.e., it *diverges*) if $n \notin A$. Evidently A is recursive if and only if both A and its complement $\bar{A}$ are r.e. There is a simple polynomial *pairing function* which encodes any pair $(x, y) \in \mathbb{N} \times \mathbb{N}$ as a number $\langle x, y \rangle \in \mathbb{N}$; we say that a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is recursive (respectively r.e.) if its image under the pairing function is recursive (respectively r.e.). A partial function $f : \mathbb{N} \to \mathbb{N}$ is recursive if its graph $\{\langle x, y \rangle : f(x) = y\}$ is a recursive subset of $\mathbb{N} \times \mathbb{N}$.

Although we have used Turing machines in our definition, an equivalent notion would be obtained if we replaced them with universal register machines, with lambda-calculus terms and reductions, or with Gurevich machines. The list of possible subsitutions is so large, in fact, and the search for a counterexample has been so fruitless, that many now accept as fact the *Church–Turing Thesis*: that the recursive sets are exactly those whose membership relation can be decided by any mechanical method at all. For this reason the recursive and recursively enumerable sets are also known as *computable* and *computably enumerable* sets, respectively.

We generalize Turing machines to *oracle Turing machines* by fixing a set $A \subseteq \mathbb{N}$, called the *oracle*, and adding a new operation to check whether a given n is in A. If a set B is computable using an oracle Turing machine with oracle A, we say B *is recursive in* A. If it can be enumerated with oracle A, we say it is *r.e. in* A. Obviously each A is recursive in itself, and as before, B is recursive in A if and only if both B and its complement $\bar{B}$ are r.e. in A. Often instead of 'recursive in' we say B *is Turing reducible to* A and write $B \leq_T A$. This relation is transitive, but not antisymmetric; for example, for any A we have $A \leq_T \bar{A}$ and $\bar{A} \leq_T A$, but of course $A \neq \bar{A}$. The *Turing degree of* A, written $\deg_T(A)$, comprises all sets $B \subseteq \mathbb{N}$ such that $A \leq_T B$ and $B \leq_T A$. The Turing degrees form a partial ordering under $\leq_T$. We call this ordering $\mathcal{D}$. If a Turing degree $\mathbf{a}$ contains an r.e. set, we call $\mathbf{a}$ a *recursively enumerable (r.e.) degree*; the partial ordering of r.e. degrees is written $\mathcal{R}$. We say a degree $\mathbf{a}$ is *recursive in* a degree $\mathbf{b}$ if some $A \in \mathbf{a}$ is recursive in some $B \in \mathbf{b}$, and that $\mathbf{a}$ is *r.e. in* $\mathbf{b}$ if some $A \in \mathbf{a}$ is r.e. in some $B \in \mathbf{b}$. It is easy to check that the choice of A does not matter in the first of these two definitions, and the choice of B does not matter in either.

0.2 The structures of $\mathcal{D}$ and $\mathcal{R}$

Not all r.e. sets are recursive. The canonical example of a properly r.e. set is the *Halting Problem*, denoted $\emptyset'$ and defined as the set of (indices of, or programs for) Turing machines which halt when given 0 as their input. By convention we write $\mathbf{0} = \deg_T(\emptyset)$ to denote the class of recursive functions, and $\mathbf{0}' = \deg_T(\emptyset')$. The Halting Problem is complete among r.e. sets in the sense that $A \leq_T \emptyset'$ for every r.e. set A. A well-known theorem of Friedberg [16] and independently Muchnik [51] states that there exist *intermediate* r.e. sets, i.e., A such that $\emptyset <_T A <_T \emptyset'$. At this point many more examples, and theorems precluding examples, of such order-theoretic property are known. See Soare [64] or [65] for an extensive, though inexhaustive, list.

Often, when constructing some specific set like the A above, our real goal is to answer coarser structural questions such as what subsets of $\mathcal{R}$ and $\mathcal{D}$ are definable, how many automorphisms they have, and in what other familiar structures they are interpretable or biinterpretable. These answers may then feed back as meta-mathematical theorems placing limits on the order-theoretic questions. An early example showing both directions is Lachlan's 1968 paper [39], which first proves that any countable distributive lattice can be embedded as an initial segment into $\mathcal{D}$, and then deduces that the first-order theory of $\mathcal{D}$ is not recursive, so that not all of its order-theoretic properties can be computed effectively.

Here we focus on questions about $\mathcal{R}$. An important tool is the *Turing jump* operator. The oracle machines with a fixed oracle A have their own, analogous version of the Halting Problem, which we denote A'. (This explains the notation $\emptyset'$, since we may think of a plain Turing machine as an oracle machine with $\emptyset$ as its oracle.) The map from A to A' is well-defined in the Turing degrees, i.e., if $\deg_T(A) = \deg_T(B)$ then $\deg_T(A') = \deg_T(B')$. The Turing jump is the inherited operator $\mathbf{a} \mapsto \mathbf{a}'$ from degrees to degrees. (This also explains the notation $\mathbf{0}'$.) The following result of Nies, Shore, and Slaman is a large step towards understanding the structure of $\mathcal{R}$.

Theorem ([52])**.** *The class* $\{\mathbf{x} \in \mathcal{R} : \mathbf{x}'' = \mathbf{a}\}$ *is definable in* $\mathcal{R}$ *(without reference to the jump operator) for each degree* $\mathbf{a}$*. (Here* $\mathbf{x}''$ *is short for* $(\mathbf{x}')'$*.)*

In Chapter 1 we present a plausible conjecture about r.e. degrees and their jumps. In conjunction with the above Theorem, this conjecture would show that $\mathcal{R}$ had no nontrivial automorphisms, thus resolving a long-standing open problem in the field. We immediately dash these hopes by showing the conjecture to be false. The key result uses a new operator called JB, for *jumps below*.

Theorem 1.1.9. Let $\mathsf{JB}(\mathbf{a}) = \{\mathbf{x}' : \mathbf{x} \leq \mathbf{a} \text{ is r.e.}\}$. If $\mathbf{c} \geq \mathbf{0}'$ is r.e. in $\mathbf{0}'$, there is a pair $\mathbf{a_0}, \mathbf{a_1}$ of distinct r.e. degrees such that $\mathbf{a}_0' = \mathbf{a}_1' = \mathbf{c}$ and $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$.

Although this defeats the attack on the automorphism problem, this theorem brings to light previously unknown properties of the Turing degrees. The operator JB, in particular, had not been studied before and deserves further investigation, and perhaps some modified version of our theorem, or of the conjecture it refutes, will lead us to a fuller understanding of $\mathcal{R}$.

0.3 Recursive model theory

Often we are interested less in the complexity of subsets of $\mathbb{N}$ or functions $\mathbb{N} \to \mathbb{N}$ than in those of countable theories, such as the first-order Peano axioms, and those of mathematical structures such as groups, rings, F-vector spaces for a particular F, and so on. Common questions include: how complicated can an isomorphic copy be, how complicated must an isomorphism between two particular copies be, how difficult is it to find a completion of a given theory, how difficult is it to build a model of a given theory, and how does it alter theorems of classical model theory when we demand that all models be computable.

In practice, we frame these questions nonetheless in terms of natural numbers by encoding a countable structure as a subset of $\mathbb{N}$. Most commonly, we encode a structure $\mathcal{A}$ with universe $A = \{a_0, a_1, \ldots\}$ as its *atomic diagram*, i.e., the set of tuples $\langle \phi, n_0, \ldots, n_k \rangle$ where ϕ is (an encoding of) a quantifier-free formula and $\mathcal{A} \models \phi(a_{n_0}, \ldots, a_{n_k})$. Since the universe $\{a_n\}_n$ can be reordered in uncountably many ways, a single structure will, in general, have encodings of uncountably many different Turing degrees.

Our contributions to recursive model theory are given in Chapters 2, 3, and 4. Chapters 2 and 3 focus on the reverse mathematics of model theory, and draw many techniques from recursive model theory. We summarize some of their results in Section 0.5 below. Chapter 4 concerns the weak truth table (wtt) degrees of models. A wtt degree is a degree of complexity based on the wtt reducibility, which is a strengthened version of $\leq_T$. Our starting point is the following dichotomy theorem of Knight:

Theorem ([37])**.** *Let $\mathcal{A}$ be a countable structure with a computable language. Then either every copy of $\mathcal{A}$ has the same Turing degree, or the degrees of its copies are upward closed in the Turing degrees.*

The main result of Chapter 4 is that when Turing degrees are replaced with wtt degrees, Knight's dichotomy admits a weakened version.

Theorem 4.3.6. Let $\mathcal{A}$ be a countable structure with a computable language. Then either there is an uncountable boldface-Π^0_1 class P such that no copy of $\mathcal{A}$ wtt-computes an element of P, or the degrees of its copies are upward closed in the wtt degrees.

We do not know whether the direct analogue of Knight's theorem holds in the wtt degrees.

0.4 Recursive sets and the arithmetical hierarchy

In Section 0.1 we listed several equivalent definitions of the recursive sets in terms of computational systems such as Turing machines and the lambda calculus. Still other characterizations make no reference to computation as one would normally think of it. A formula in the language of first-order arithmetic $(0, 1, +, \cdot, <)$ is Σ^0_0 if every quantifier in it is bounded, i.e., is of the form $(\exists x < a)$ or $(\forall x < a)$; a formula is Σ^0_1 if it is of the form $(\exists x)\phi$ where ϕ is bounded; a subset of $\mathbb{N}$ is Σ^0_1 if it is defined by a Σ^0_1 formula. A remarkable theorem of Post characterizes the recursive and r.e. sets in these terms: A set A is r.e. if and only if A is defined by a Σ^0_1 formula, and furthermore A is recursive if and only if A is Δ^0_1, i.e., both A and its complement $\bar{A}$ are Σ^0_1. (This was later sharpened, as the main step in solving Hilbert's tenth problem, to purely $\exists$ formulas with no bounded quantifiers.)

Through Post's theorem we could develop the theory of recursive and r.e. functions in terms of elementary number theory. We normally do not do this—Turing machines are easier to reason about—but the correspondence finds practical use when extended to Σ^0_n for higher n. For each $n \geq 2$, a set A is Σ^0_{n+1} if $n \in A \iff (\exists s)\langle n, s\rangle \notin B$ for some Σ^0_n set B. A set is Δ^0_n if both it and its complement are Σ^0_n. A set is Δ^0_2 if and only if it is recursive in the Halting Problem $\emptyset'$; it is Σ^0_2 if and only if it is r.e. in $\emptyset'$; it is Δ^0_3 if and only if it is recursive in $\emptyset''$, the 'Halting Problem's Halting Problem'; and so on. This *arithmetical hierarchy* classifies many decision problems in a way that is both intuitive and amenable to proof-theoretic analysis: the induction axioms of first-order Peano arithmetic are themselves stratified along these lines of complexity. This is important in reverse mathematics, and especially in our work in Chapter 3.

0.5 Reverse mathematics

Reverse mathematics is a foundational programme initiated by H. Friedman in [17, 18] and developed extensively by Simpson and many others. Simpson's book [63] is at once the broadest introduction and the most complete reference. The goal is to determine, given a mathematical theorem, exactly what set-existence axioms are necessary and sufficient to prove it in the setting of second-order arithmetic. (The namesake *reversal* is the preferred method of establishing necessity: turn the tables and use the theorem itself to prove the axiom.) This can be motivated through a series of three observations.

The first, which justifies the chosen setting, is that the vast majority of ordinary mathematics can be formalized and proved in the theory of second-order arithmetic. Here *ordinary mathematics* is an informal term understood to encompass most of the mathematical results found in a typical textbook or journal, but to exclude those few foundational or set-theoretic topics, such as the study of higher infinities, whose objects are by design too large or too complicated for second-order arithmetic to handle.

The idea of encoding objects as sets is older than recursion theory—consider, for instance, the representation of real numbers by Dedekind cuts—and the idea of studying their complexity is scarcely younger. Inasmuch as an object's complexity is concerned, reverse mathematics is just one of several parallel investigations, alongside recursive mathematics (including recursive model theory, outlined in Section 0.3 above, but also recursive analysis and recursive algebra), constructive mathematics (including Bishop-style constructive analysis), and effective descriptive set theory. The second observation, fundamental to reverse mathematics but largely irrelevant to these sister areas, is that the complexity of two theorems can often be compared, even if at first blush they seem completely unrelated. For example, by coding finite sets of formulas as rational numbers, we can use the compactness (in the sense of open covers) of the unit interval $[0, 1]$ to prove the completeness theorem for propositional logic. Thus theorems and other well-formed statements from many areas can be placed into a single partial ordering based on their comparative strength.

The third and most striking observation is that most theorems appear to fall into one of five classes of mutual reducibility, ordered linearly in terms of strength. Known colloquially as the Big Five, they are, from weakest to strongest: RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. Each of these classes named for one of one of its constituent statements, selected by Friedman, that distills every other element, or the idea behind their proofs, to a simple combinatorial or logical axiom. For example, WKL_0 is named for Weak König's Lemma, which states: Every infinite binary tree has an infinite path. It is not difficult to check informally that WKL_0 contains both the theorem stating that $[0, 1]$ is compact, and the completeness theorem for propositional logic.

Nevertheless, some theorems are known to exist outside the Big Five, and to be equivalent instead to an induction axiom, for instance, or to a combinatorial statement such as the countable Ramsey theorem, or to some perturbation such as Weak Weak König's Lemma (which is simply Weak König's Lemma for positive-measure trees). Interest in non-Big-Five theorems has grown in recent years, following a paper of Cholak, Jockusch, and Slaman [9] which correctly hinted that Ramsey's theorem might be of this variety, and subsequent work of Hirschfeldt and Shore [29] exploring natural statements strictly weaker than Ramsey's theorem.

If a theorem admits more than one proof, say one by induction and another by a compactness argument, it may in rare cases be equivalent to a disjunction of two principles. The first such example was found in the field of dynamical systems by Friedman, Simpson, and Yu [20] and shown to be equivalent to $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$ (where $\mathsf{I}\Sigma^0_2$ denotes the induction axiom for Σ^0_2-definable sets). Our own contributions to reverse mathematics, comprising Chapters 2 and 3 below, focus on statements from basic model theory. This topic has been looked at before by other authors, for example: [25, 28, 30, 40]. In Chapter 2 we focus on the Ryll-Nardzewski Theorem, Vaught's no-two-model theorem, and Ehrenfeucht's three-model theorem; in Chapter 3 we compare a number of definitions and existence theorems for homogeneous and saturated models. While most of our results say that a certain theorem falls a certain Big Five equivalence class, we also obtain some rare or novel classes, to wit: Proposition 3.2.13 gives a statement weaker than $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$, Theorem 3.2.24 gives a family of statements equivalent to $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$, and, most strikingly, Theorems 2.2.12 and 3.2.28 give a family equivalent to $\neg\mathsf{WKL}_0 \vee \mathsf{ACA}_0$.

The results of Chapter 2 have recently been extended by Fokina, Li, and Turetsky [15].

CHAPTER 1

RECURSIVELY ENUMERABLE SETS AND THE QUESTION OF RIGIDITY

This chapter is joint work with Richard A. Shore. It has been accepted, in a revised form, for publication in the Notre Dame Journal of Formal Logic [6].

1.1 Introduction

The general setting for this paper is the study of the relationships between a degree $\mathbf{a}$ and its jump $\mathbf{a}'$ and, more generally, between $\mathbf{a}$ and the degrees $REA(\mathbf{a})$, i.e. those recursively enumerable in and above $\mathbf{a}$. The question that concerns us here is to what extent a degree, or more specifically, an r.e. degree $\mathbf{a}$ is determined by the jumps of, or degrees REA in, degrees $\mathbf{x}$ near $\mathbf{a}$. In particular, we were motivated by a conjecture about these relationships that would have implied the rigidity of the r.e. degrees, $\mathbf{R}$. The conjecture was inspired by the hope of combining two important results. One by Soare and Stob [66] tells us that, under certain conditions, we can find degrees REA in a given $\mathbf{a}$ but not in another $\mathbf{b}$. The second is the constellation of Jump Interpolation Theorems of Robinson [56]. These theorems generally say that any simple statement about the ordering of r.e. degrees and and their jumps (e.g. extension-of-embeddings results) not shown false by an obvious property of the r.e. degrees and their jumps can be realized. We give specific versions of these results that we need for our analysis.

Theorem 1.1.1 (Soare–Stob)**.** *If $\mathbf{0} < \mathbf{a} \in \mathbf{R}$, then there is a degree $\mathbf{c}$ which is $REA(\mathbf{a})$ but not r.e.*

Theorem 1.1.2 (Robinson Jump Interpolation)**.** *If $\mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathbf{R}$, $\mathbf{e} \not\leq \mathbf{c} < \mathbf{d}$, $\mathbf{z} \geq \mathbf{c}'$ and is $REA(\mathbf{d})$, then there is an $\mathbf{f} \in \mathbf{R}$ with $\mathbf{c} < \mathbf{f} < \mathbf{d}$, $\mathbf{e} \not\leq \mathbf{f}$, and $\mathbf{f}' = \mathbf{z}$.*

Our thought was to combine and extend these ideas (and their standard relativizations) so as to characterize some r.e. degrees in terms of the jumps of degrees below them or equivalently (see Lemma 1.2.2) in terms of the degrees r.e. in them and above $\mathbf{0}'$. If we could do this for enough r.e. degrees, we knew that we could prove the rigidity of $\mathbf{R}$. We begin with the definition of our primary object of study.

Definition 1.1.3. If $\mathbf{a}$ is a Turing degree, the class $\mathsf{JB}(\mathbf{a})$, *jumps below* $\mathbf{a}$, consists of the jumps of degrees below $\mathbf{a}$:

$$\mathsf{JB}(\mathbf{a}) = \{\mathbf{x}' : \mathbf{x} \leq \mathbf{a}\}.$$

We will see in Lemma 1.2.2 that, by Theorem 1.1.2, if $\mathbf{a} \in \mathbf{R}$ then in this definition we can consider just those $\mathbf{x} \leq \mathbf{a}$ which are also r.e. and that $\mathsf{JB}(\mathbf{a}) = \{\mathbf{z} : \mathbf{0}' \leq \mathbf{z} \ \& \ \mathbf{z} \in \mathbf{REA}(\mathbf{a})\}$. (As usual, we use $\mathbf{REA}(\mathbf{a})$ to denote the class of degrees $REA(\mathbf{a})$.)

As Soare and Stob [66] point out, relativizing Theorem 1.1.1 to any incomplete *high* degree $\mathbf{h}$ (i.e. $\mathbf{h}' = \mathbf{0}''$) and taking $\mathbf{0}'$ to play the role of $\mathbf{a}$, one sees that, for any incomplete high degree $\mathbf{h}$, there is a $\mathbf{c}$ $REA(\mathbf{0}')$ which is not $REA(\mathbf{h})$. Thus $\mathsf{JB}(\mathbf{0}') \neq \mathsf{JB}(\mathbf{a})$ for any incomplete r.e. $\mathbf{a}$ and so $\mathbf{0}'$ is determined within $\mathbf{R}$ by $\mathsf{JB}(\mathbf{0}')$. (If $\mathbf{a}$ is not high, then it is trivial that $\mathsf{JB}(\mathbf{0}') \neq \mathsf{JB}(\mathbf{a})$ as $\mathbf{0}'' \notin \mathsf{JB}(\mathbf{a})$ but, of course, $\mathbf{0}'' \in \mathsf{JB}(\mathbf{0}')$.) Our goal was to extend this to other degrees. Our hope was that we could characterize enough r.e. degrees $\mathbf{a}$ in terms of $\mathsf{JB}(\mathbf{a})$ to provide an automorphism basis that would be fixed under all automorphisms of $\mathbf{R}$ and use this to prove its rigidity.

A slightly different relativized version of Theorem 1.1.1 is as follows:

Corollary 1.1.4 (Soare and Stob)**.** *For any r.e.* $\mathbf{a}$ *and* $\mathbf{b}$ *with* $\mathbf{a} \not\leq \mathbf{b}$*, there is a* $\mathbf{c}$ $REA(\mathbf{a})$ *which is not* $REA(\mathbf{b})$*.*

Proof. If $\mathbf{b} < \mathbf{a}$ then this is the straightforward relativization of Theorem 1.1.1 as just used for $\mathbf{h} < \mathbf{0}'$. Otherwise $\mathbf{a}$ itself is the desired $\mathbf{c}$. □

The theme of the Robinson Interpolation Theorems is that anything not ruled out by simple relations between degrees and their jumps should be realizable. Along these lines we hoped to prove that the witness $\mathbf{c}$ in Corollary 1.1.4 (which is to be REA in $\mathbf{a}$ but not in $\mathbf{b}$) could be taken to lie between $\mathbf{0}'$ and $\mathbf{a}'$ at least for many degrees. Clearly this is not possible if these degrees are low, i.e. $\mathbf{a}' = \mathbf{b}' = \mathbf{0}'$ but there are many other candidates. What constitutes "enough degrees" here is driven by the desire to get an automorphism basis for $\mathbf{R}$ whose elements $\mathbf{a}$ would be determined by the degrees $REA(\mathbf{a})$ and above $\mathbf{0}'$, i.e. by $\mathsf{JB}(\mathbf{a})$. In particular, we state our original Conjecture for the class $\mathbf{H}_2 = \{\mathbf{x} \in \mathbf{R} | \mathbf{x}'' = \mathbf{0}'''\}$ of *high*$_2$ r.e. degrees which is known to be an automorphism basis for $\mathbf{R}$ (see §1.4).

Conjecture 1.1.5. *If $\mathbf{a}, \mathbf{b} \in \mathbf{H}_2$ and $\mathbf{a} \not\leq \mathbf{b}$, then there is a $\mathbf{c} \geq \mathbf{0}'$ which is $REA(\mathbf{a})$ but not $REA(\mathbf{b})$. As usual this should also be true relativized to any $\mathbf{x}$.*

In §1.4 we show that this conjecture would imply the rigidity of $\mathbf{R}$ and so proving it would have answered many of the most important questions about $\mathbf{R}$. Of course, if $\mathbf{a}$ and $\mathbf{b}$ have different jumps then, trivially, $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ so the real questions only arise when $\mathbf{a}' = \mathbf{b}'$. Known results can be used to show that we can at times distinguish between some $\mathbf{a}$ and $\mathbf{b}$ (with the same jump) by distinguishing between $\mathsf{JB}(\mathbf{a})$ and $\mathsf{JB}(\mathbf{b})$. The simplest examples follow easily from Theorems 1.1.1 and 1.1.2:

Proposition 1.1.6. *If $\mathbf{h} \in \mathbf{R}$ is high then there is an incomplete high r.e. $\mathbf{g}$ with $\mathsf{JB}(\mathbf{h}) \neq \mathsf{JB}(\mathbf{g})$. In fact, we may find such a $\mathbf{g}$ which is incomparable with $\mathbf{h}$.*

Proof. As mentioned above, relativizing Theorem 1.1.1 to $\mathbf{h}$ and applying it to $\mathbf{0}'$ gives a $\mathbf{c}$ $REA(\mathbf{0}')$ and not $REA(\mathbf{h})$. Applying Theorem 1.1.2 (or just the jump theorem of Sacks [58] gives us an r.e. $\mathbf{k}$ such that $\mathbf{k}' = \mathbf{c}$. Of course, as $\mathbf{k}$ is not high, $\mathbf{h} \not\leq \mathbf{k}$. Applying Theorem 1.1.2 again gives us an incomplete high r.e. $\mathbf{g} \geq \mathbf{k}$ with $\mathbf{h} \not\leq \mathbf{g}$. It is now clear that $\mathbf{c} \in \mathsf{JB}(\mathbf{g})$ but $\mathbf{c} \notin \mathsf{JB}(\mathbf{h})$ and so $\mathbf{g} \not\leq \mathbf{h}$ as well. □

We can extend and then apply a result of Arslanov, Lempp and Shore [1, Proposition 1.13] to get the same result for the high$_2$ r.e. degrees.

Proposition 1.1.7 (Arslanov, Lempp and Shore)**.** *If* $\mathbf{c} < \mathbf{h}$ *are r.e.,* $\mathbf{c}$ *is low, i.e.* $\mathbf{c}' = \mathbf{0}'$*, and* $\mathbf{h}$ *is high, then there is an* $\mathbf{a} < \mathbf{h}$ *which is* $REA(\mathbf{c})$ *but not r.e.*

Corollary 1.1.8. *If* $\mathbf{x} \in \mathbf{R}$ *is high*$_2$ *then there is an r.e. degree* $\mathbf{g}$ *with* $\mathbf{x}' = \mathbf{g}'$ *such that* $\mathsf{JB}(\mathbf{x}) \neq \mathsf{JB}(\mathbf{g})$*. In fact, we may find such a* $\mathbf{g}$ *which is incomparable with* $\mathbf{x}$*.*

Proof. First note that the Proposition can be improved to allow $\mathbf{c}$ to be low_2, i.e. $\mathbf{c}'' = \mathbf{0}''$: Given such a $\mathbf{c}$ and a high $\mathbf{h} > \mathbf{c}$ apply Theorem 1.1.2 to get an r.e. $\mathbf{d}$ with $\mathbf{d} < \mathbf{c} < \mathbf{h}$ and $\mathbf{d}' = \mathbf{c}'$. Now relativize the Proposition to $\mathbf{d}$ and note that $\mathbf{c}$ is low relative to $\mathbf{d}$ while $\mathbf{h}$ is still high relative to $\mathbf{d}$ as $\mathbf{h}' = \mathbf{0}'' = \mathbf{d}''$. Thus we have an $\mathbf{a}$ which is $REA(\mathbf{c})$ but not r.e. in $\mathbf{d}$ and so certainly not r.e.

Next, relativize this extension of the Proposition to our given $\mathbf{x}$ and apply it to $\mathbf{0}'$ (as $\mathbf{c}$) and $\mathbf{x}'$ (as $\mathbf{h}$). (As $\mathbf{x} \in \mathbf{H}_2$, $\mathbf{0}'$ is low$_2$ relative to $\mathbf{x}$ while $\mathbf{x}'$ is obviously high relative to $\mathbf{x}$.) This gives us an $\mathbf{a}$ with $\mathbf{0}' < \mathbf{a} < \mathbf{x}'$ such that $\mathbf{a}$ is $REA(\mathbf{0}')$ but not r.e. in $\mathbf{x}$. Now argue as for Proposition 1.1.6 using Theorem 1.1.2: First one gets an r.e., $\mathbf{k} \not\geq \mathbf{x}$ with $\mathbf{k}' = \mathbf{a}$. Then one gets an r.e. $\mathbf{g} \not\geq \mathbf{x}$ with $\mathbf{g} \geq \mathbf{k}$ and $\mathbf{g}' = \mathbf{x}'$ so that $\mathbf{a} \in \mathsf{JB}(\mathbf{g})$ but $\mathbf{a} \notin \mathsf{JB}(\mathbf{x})$. (Of course, this also implies that $\mathbf{g} \not\leq \mathbf{x}$.) □

While these last results show that there are many degrees such that we can distinguish between them in terms of the JB operator, the main result of this paper is to exhibit (regrettably) a rather strong failure of any possible characterization of the r.e. degrees $\mathbf{a}$ in any particular *jump class*, i.e. those $\mathbf{a}$ with $\mathbf{a}' = \mathbf{c}$ for any $\mathbf{c}$ $REA(\mathbf{0}')$ based simply on $\mathsf{JB}(\mathbf{a})$.

Theorem 1.1.9 (Main Theorem)**.** *If* $\mathbf{c}$ *is* $REA(\mathbf{0}')$, *then*

1. *there is a pair* $\mathbf{a}_0, \mathbf{a}_1$ *of r.e. degrees such that* $\mathbf{a}_0' = \mathbf{a}_1' = \mathbf{c}$, $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$ *and* $\mathbf{a}_0|\mathbf{a}_1$*; and*
2. *there is a pair* $\mathbf{b}_0, \mathbf{b}_1$ *of r.e. degrees such that* $\mathbf{b}_0' = \mathbf{b}_1' = \mathbf{c}$, $\mathsf{JB}(\mathbf{b}_0) = \mathsf{JB}(\mathbf{b}_1)$ *and* $\mathbf{b}_0 < \mathbf{b}_1$.

Part 1 of the Main Theorem is proved in §1.2, by an argument extending the usual $0''$ tree proof of the Sacks jump theorem. Part 2 we do not prove in full; instead, in §1.3 we outline how to modify the proof from §1.2 to get degrees which are comparable instead of incomparable.

Before presenting the proofs, we mention a couple of the natural questions raised by these results and proofs and discuss some methodological issues.

Question 1.1.10. Are there any incomplete r.e. degrees $\mathbf{a}$ characterized by $\mathsf{JB}(\mathbf{a})$, i.e. such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ for any r.e. $\mathbf{b} \neq \mathbf{a}$? If so, are there enough to constitute an automorphism base for $\mathbf{R}$ and could one then prove its rigidity?

In the other direction, there are several possible strengthenings of the Main Theorem. We mention one that would provide a negative answer to the previous question.

Question 1.1.11. Is there, for every r.e. $\mathbf{a}$, an r.e. $\mathbf{b} \neq \mathbf{a}$ such that $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$?

There are many possible variations on these questions some of which we discuss in §1.5 along with a couple of consequences of previous work which bear on them.

We also want to remark here on an unusual aspect of the construction for Theorem 1.1.9. While in several ways, it is quite similar to the usual proof of the Sacks jump theorem, satisfying the requirements to make $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$ seems to require an unusual organization of the priority tree (certainly unusual for $0''$ constructions such as the Sacks jump theorem). What would seem to be individual requirements for this goal (the $\mathcal{R}_{e,i}$ below) are divided up into infinitely many subrequirements (the $\mathcal{R}_{e,i,n}$ below). These subrequirements are spread across the tree (rather than, for example, along the paths below a node assigned to $\mathcal{R}_{e,i}$ as is often the case). In combination with other requirements, the subrequirements (to the right of the true path) can interfere with each other with the possible outcome of subverting the final satisfaction of the basic requirement (even along the true path). Our solution is to change the priority of these subrequirements in a dynamic way that depends on the actions of nodes to their left. In particular, nodes on the priority tree are assigned different requirements at different stages of the construction. While this is common in $0'''$ constructions it is unusual in $0''$ ones (as ours is). Moreover, the nodes are assigned requirements in a way that does not depend solely on the outcomes along the path leading to the node (and possibly external approximation procedures as well). This seems unusual (if not unique) even for $0'''$ arguments.

The first example of which we are aware of changes in the priority of requirements in a somewhat similar way occurs in some cases of the minimal pair construction in α-recursion theory (Shore [61]). Another relatively early result that has variable priority assignments is Theorem 2 of Jockusch and Soare [33]. They construct a low linear order with a predicate for infinitely far apart not isomorphic to a recursive one. An unusual construction with a requirement analogous to our $\mathcal{R}_{e,i}$ (to make various sets have r.e. degree) occurs in the $0'''$ construction for Theorem 3.1 of Arslanov, Lempp and Shore [1]. (We use this result below in Proposition 1.5.5.) The unusual procedure employed there is allowing nodes to act when they are to the left of the true path. A similar idea would probably work here as well but our construction while unusual in a different way seems simpler in this case.

We discuss some consequences of this unusual construction for a reverse mathematical analysis of our theorem in §1.5.

1.2 Proving part (1) of the theorem

Suppose $\mathbf{c}$ is $REA(\mathbf{0}')$ and choose a representative $C \in \mathbf{c}$. We may fix an r.e. set D such that, for all n, the n-th column $D^{[n]} = \{x : \langle n, x\rangle \in D\}$ is an initial segment of ω, finite if $n \in C$, and equal to ω if $n \notin C$. We assume that no element s enters D before stage s. We will use D to build a pair A_0, A_1 of sets such that $\mathbf{a}_0 = \deg(A_0)$ and $\mathbf{a}_1 = \deg(A_1)$ are as required by the theorem. Our argument closely follows the usual pattern of a $0''$ tree construction, with the peculiar feature that the assignment of requirements to nodes is allowed to vary from stage to stage.

Before we can specify our requirements, we need some more notation, and we need a way of characterizing $\mathsf{JB}(\mathbf{a})$ suitable for a priority construction. We use the following representation for relative enumerations:

Definition 1.2.1. If V is an r.e. set with enumeration $(V_s)_{s\in\omega}$ and A is any set, then V^A is the set $\{n : (\exists \sigma \subseteq A)\langle n, \sigma\rangle \in V\}$. For each s, we define V_s^A likewise, with V_s in place of V.

Note that the class of sets r.e. in A is equal to $\{W_e^A | \ e \in \omega\}$.

Lemma 1.2.2. *For every degree* $\mathbf{y}$ *and every r.e. degree* $\mathbf{a}$ *with representative* $A \in \mathbf{a}$*, the following are equivalent:*

1. $\mathbf{y} = \mathbf{x}'$ *for some* $\mathbf{x} \leq \mathbf{a}$*.*

2. $\mathbf{y} = \mathbf{x}'$ *for some r.e.* $\mathbf{x} \leq \mathbf{a}$.

3. $\mathbf{y} \geq \mathbf{0}'$ *and* $\mathbf{y}$ *is* $REA(\mathbf{a})$.

4. $\mathbf{y} = \deg(W_e^A \oplus 0')$ *for some* e.

Proof. The implications $(2 \Rightarrow 1)$ and $(3 \Rightarrow 4)$, and $(4 \Rightarrow 3)$ are immediate. The implication $(1 \Rightarrow 3)$ follows from the monotonicity of the jump operator and the fact that if $X \leq_T A$ and Y is r.e. in X then Y is r.e. in A. The final implication $(3 \Rightarrow 2)$ follows from Theorem 1.1.2. □

The equivalence between 1 and 4 implies in particular that $\mathsf{JB}(\mathbf{a}) = \{\deg(W_e^A \oplus 0')|e \in \omega\}$. It is this characterization that we use to frame our requirements.

1.2.1 Requirements

We begin by listing four basic goals for the construction.

- $A_0 \not\leq_T A_1$ and $A_1 \not\leq_T A_0$
- $C \leq_T A_0', A_1'$
- $A_0', A_1' \leq_T C$
- For every $e \in \omega$ and $i \in \{0,1\}$, there is an r.e. set $V_{e,i}$ such that $W_e^{A_{1-i}} \oplus 0' \equiv_T V_{e,i}^{A_i} \oplus 0'$.

The first three goals are self-explanatory, while the fourth guarantees through Lemma 1.2.2 that $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$. Hence a construction of any $A_i, V_{e,i}$, $e \in \omega$, $i \in \{0,1\}$ meeting these goals will constitute a proof of the Main Theorem, part 1. We represent the first, second and fourth goals as requirements named with the letters $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{R}$, respectively. The third we do not capture directly as a requirement, although in the end it is satisfied by the $\mathcal{R}$ strategy as well (Proposition 1.2.10 below).

The $\mathcal{N}$ and $\mathcal{P}$ requirements are as in the usual proofs of the Friedberg-Muchnik and Sacks jump theorems, respectively. Namely, we ensure that $A_i \not\leq_T A_{1-i}$ by using infinitely many diagonalization requirements:

$\mathcal{N}_{e,i} : A_i \neq \Phi_e^{A_{1-i}}$, for all $e \in \omega$, $i \in \{0, 1\}$.

We ensure that $C \leq_T A_0', A_1'$ by using infinitely many thickness requirements:

$\mathcal{P}_e : A_0^{[e]} =^* D^{[e]}$ and $A_1^{[e]} =^* D^{[e]}$. (Here $X =^* Y$ means that X, Y differ only by a finite set.)

We attack the fourth goal by breaking it up, for each $e \in \omega$ and each $i \in \{0, 1\}$, into infinitely many requirements $\mathcal{R}_{e,i,n}$, with n ranging over ω. The idea is to construct a single r.e. set $V_{e,i}$ so that $W_e^{A_{1-i}}$ contains a number n if and only if $V_{e,i}^{A_i}$ contains $\langle n, m\rangle$ for some $m = m_{e,i,n}$ which can be computed by $0'$ (so that $W_e^{A_{1-i}} \leq_T V_{e,i}^{A_i} \oplus 0'$), while keeping the question of whether $\langle n, k\rangle$ is in $V_{e,i}^{A_i}$ easy to answer (given m) for all $k \neq m$ (so that $V_{e,i}^{A_i} \leq_T W_e^{A_{1-i}} \oplus 0'$). The formal requirements, and a lemma showing that they suffice to guarantee that $W_e^{A_{1-i}} \oplus 0' \equiv_T V_{e,i}^{A_i} \oplus 0'$, are as follows:

$\mathcal{R}_{e,i,n}$: There is an r.e. set $V_{e,i}$, which does not depend on n, and a number $m_{e,i,n}$ computable uniformly in n from $0'$ such that $n \in W_e^{A_{1-i}} \Leftrightarrow \langle n, m_{e,i,n}\rangle \in V_{e,i}^{A_i}$. Furthermore, $k < m_{e,i,n}$ implies $\langle n, k\rangle \in V_{e,i}^{A_i}$, and $k > m_{e,i,n}$ implies $\langle n, k\rangle \notin V_{e,i}^{A_i}$.

Lemma 1.2.3. *Fix e and i. If $\mathcal{R}_{e,i,n}$ is met for all n, then $W_e^{A_{1-i}} \oplus 0' \equiv_T V_{e,i}^{A_i} \oplus 0'$.*

Proof. To compute whether n is in $W_e^{A_{1-i}}$, first use $0'$ to find $m_{e,i,n}$, and then check whether $\langle n, m_{e,i,n}\rangle$ is in $V_{e,i}^{A_i}$. This shows that $W_e^{A_{1-i}} \leq_T V_{e,i}^{A_i} \oplus 0'$. To compute whether $\langle n, k\rangle$ is in $V_e^{A_i}$, first use $0'$ to find $m_{e,i,n}$, and compare it with k: if $k < m_{e,i,n}$ then the answer is yes, if $k > m_{e,i,n}$ then the answer is no, and if $k = m_{e,i,n}$ then it is enough to check whether n is in $W_e^{A_{1-i}}$. Hence $V_{e,i}^{A_i} \leq_T W_e^{A_{1-i}} \oplus 0'$. $\square$

1.2.2 Notation and bookkeeping

The tree of nodes. We assign requirements to nodes on a tree as in a typical $\mathbf{0}''$ priority argument, except that the assignment of requirements to nodes is allowed to change from stage to stage. The tree itself is $\{\mathtt{0},\mathtt{1}\}^{<\omega}$, ordered lexicographically with $\mathtt{0} <_{\text{lex}} \mathtt{1}$ as usual. The $\mathcal{P}$ requirements have two possible outcomes, $\mathtt{0}$ and $\mathtt{1}$, representing an infinitary and a finitary action, respectively. The $\mathcal{R}$ and $\mathcal{N}$ requirements have only one outcome, $\mathtt{0}$.

The accessible path δ_s. At each stage s, we specify a node $\delta_s \in \{\mathtt{0},\mathtt{1}\}^s$. (The precise construction of δ_s is presented in §1.2.3 below.) We say that δ_s and each of its initial segments $\alpha \subseteq \delta_s$ are *accessible at stage s*. No other node is accessible at stage s.

Restraints $r_{\alpha,s}$ and $r_{<\alpha,s}$. At each stage s, each node $\alpha \in \{\mathtt{0},\mathtt{1}\}^{<\omega}$ places a *restraint* $r_{\alpha,s}$ limiting the possible actions of nodes that are lexicographically greater than α. For each α, the initial value is $r_{\alpha,0} = 0$. If α is not accessible, then $r_{\alpha,s} = r_{\alpha,s-1}$. Otherwise, $r_{\alpha,s}$ is as specified in §1.2.3 below. For each α and s, we use $r_{<\alpha,s}$ to denote $\max\{r_{\beta,s} : \beta <_{\text{lex}} \alpha\}$. Notice that at every stage s cofinitely many $r_{\alpha,s}$ are equal to zero, and every $r_{<\alpha,s}$ is finite.

Assigning requirements to nodes. The $\mathcal{R}$ requirements are sensitive to injury because a single set $V_{e,i}$ is shared across all nodes assigned an $\mathcal{R}_{e,i,n}$ requirement, and distinct $\mathcal{R}_{e,i,n}$ with the same e and i may each be adding elements to $V_{e,i}$. These elements cannot subsequently be removed, as $V_{e,i}$ is r.e. Thus a version of an $\mathcal{R}_{e,i,n}$ requirement can act far to the right of the true path (which is defined as usual in §1.2.4) and we may later have to react to some injury from a node to its left (but still to the right of the true path) by making the only correction we can, i.e. changing the value of $m_{e,i,n}$. Allowing this to happen infinitely often will send $m_{e,i,n}$ to infinity and ruin our coding procedure for computing $W_e^{A_{1-i}}(n)$. One appropriate response is to increase the priority of the $\mathcal{R}_{e,i,n}$ requirements, relative to those that injured them, each time this happens; the countervailing constraint is the obvious one that we must eventually deal with all the requirements (on the true path) and so cannot increase the priority of all the $\mathcal{R}_{e,i,n}$ arbitrarily. The solution is to increase the priority of the $\mathcal{R}_{e,i,n}$ requirements in a controlled way that allows other requirements to act as well along the true path. We do this by assigning the requirements dynamically, i.e. by a scheme that depends on the stage s. We could define an assignment simultaneously with the full construction that depends directly on the nodes accessible at s and the actions taken (injuries sustained) at stages less than or equal to s. While that might produce a more intuitive definition (given that one already understood the construction), we instead give a simple (if uninformative) definition that is independent of the construction's details and that uses a counting argument to allow the priority of an $\mathcal{R}_{e,i,n}$ requirement to increase while still leaving room for the other requirements on the true path. This makes both the assignment of requirements to nodes on the tree and the eventual verifications significantly simpler.

The precise assignment scheme is as follows. First, fix some recursive list of all the $\mathcal{R}$ requirements and a second recursive list of all the $\mathcal{P}$ and $\mathcal{N}$ requirements. At stage s, we assign a requirement to each node $\alpha \in \{\mathtt{0},\mathtt{1}\}^{<\omega}$ by recursion on its initial segments. Let u be the number of proper initial segments of α assigned an $\mathcal{R}$ requirement at stage s, and let v be the number of nodes $\beta \leq_{\text{lex}} \alpha$ which have been accessible at any stage $t \leq s$. If $u < v/2$, then assign to α the $(u+1)$-th $\mathcal{R}$ requirement; otherwise, assign to α the next unused (i.e. the $(|\alpha| - u + 1)$-th) requirement from the $\mathcal{P}, \mathcal{N}$ list.

Conventions for V^A, *use*, and $\langle \cdot, \cdot \rangle$. The use of a convergent computation

$\Phi_e^A(x)$ or $\Phi_{e,s}^A(x)$ is the least $u \leq s$ such that $\Phi_{e,u}^{A\upharpoonright u}(x)\downarrow$. If V^A is as in Definition 1.2.1, we identify V^A with its characteristic function as usual; if $V^A(n) = 1$, the *use* of $V^A(n)$ is the shortest $\sigma \subseteq A$ such that $\langle n, \sigma\rangle \in V$. We do not define a use for $V^A(n) = 0$. We follow the convention that $\Phi_{e,s}(x)\downarrow$ only if $x < s$, and a V_s in an r.e. approximation $(V_s)_s$ may contain n only if $n < s$. The pairing function $\langle x, y\rangle$ is recursive and increasing in each coordinate. Each binary string σ is naturally identified with a natural number through its binary expansion; this number grows monotonically with the length-lexicographic ordering. The pairing function is left-associative, so we may write $\langle n, m, \sigma\rangle$ for $\langle\langle n, m\rangle, \sigma\rangle$.

α-believable computations. Fix any $\alpha \in \{\mathtt{0},\mathtt{1}\}^{<\omega}$, and suppose $\Phi_{e,s}^{A_{1-i,s}}(x)\downarrow$ with use u. We say this computation is an *α-believable computation at stage s* if for every $\mathcal{P}_j$ which is assigned to an initial segment $\beta \subseteq \alpha$ at stage s with outcome $\alpha(|\beta|) = \mathtt{0}$, we have

$$\{k \in A_{1-i,s}^{[j]} : r_{<\alpha,s} \leq \langle j, k\rangle \leq u - 1\} = \{k : r_{<\alpha,s} \leq \langle j, k\rangle \leq u - 1\}$$

where $[x, y]$ denotes a closed interval in ω.

Now fix $\alpha \in \{\mathtt{0},\mathtt{1}\}^{<\omega}$ and suppose $n \in W_{e,s}^{A_{1-i,s}}$ by $\langle n, \sigma\rangle \in W_{e,s}$ with $\sigma \subseteq A_{1-i,s}$. We call this enumeration an *α-believable computation at stage s* if, for all j as above,

$$\{k \in A_{1-i,s}^{[j]} : r_{<\alpha,s} \leq \langle j, k\rangle \leq |\sigma| - 1\} = \{k : r_{<\alpha,s} \leq \langle j, k\rangle \leq |\sigma| - 1\}$$

1.2.3 The basic strategies and outcomes; defining δ_s

Suppose $k < s$ is fixed, $\alpha = \delta_s \upharpoonright k$, and α is assigned the requirement $\mathcal{Q}$ at stage s. Our strategy for α determines any changes made by α to A_0, A_1, $V_{e,0}$, $V_{e,1}$, or $m_{e,i,n}$ at stage s, the restraint $r_{\alpha,s}$, and the outcome $\delta_s(k)$, and with it, if $k < s-1$, the next accessible node $\alpha^\frown\delta_s(k)$. We say that a node α acts at stage s if and only if its strategy changes one of A_0, A_1, $V_{e,0}$, $V_{e,1}$, $m_{e,i,n}$, or $r_{\alpha,s} \neq r_{\alpha,s-1}$ at stage

s. If the construction does not explicitly change one of these sets or variables at stage s, then it takes the same value as at stage $s-1$. Here are the strategies:

If $\mathcal{Q} = \mathcal{P}_e$: If this is the first time α has been accessible or if no new element has entered $D_s^{[e]}$ since last time α was accessible, do nothing; the outcome is the finitary outcome `1`. Otherwise, add to $A_0^{[e]}$ and $A_1^{[e]}$ all k such that $r_{<\alpha,s} \leq \langle e, k\rangle$ and $k \leq s$. In this case, the outcome is `0`. [The intention is, as usual, that $A_i^{[e]}$ will be finite if $D^{[e]}$ is finite, and cofinite if $D^{[e]}$ is ω. We add whole intervals at once to make it easier to determine when, and in what way, this action injures lower-priority requirements.]

If $\mathcal{Q} = \mathcal{N}_{e,i}$: Check whether there is an x in the interval $r_{<\alpha,s} < x < s$ such that

i. $x \neq \langle j, k\rangle$ for all $j < |\alpha|$ and all k; and

ii. $\Phi_{e,s}^{A_{1-i,s}}(x) \downarrow = y$ by an α-believable computation, where either $y = 0$, or $y \neq A_{i,s}(x)$.

If there is no such x, do nothing. Otherwise, take the least x which minimizes the use of the convergent computation $\Phi_{e,s}^{A_{1-i,s}}(x)$, and consider the value of y from condition (ii). If $y = 0$ and x is not in $A_{i,s}$, add x to A_i; otherwise, do not change A_i. In either case, set the restraint $r_{\alpha,s}$ to equal the use of the computation $\Phi_{e,s}^{A_{1-i,s}}(x)$. [The minimization is to guarantee that after the requirement has been "permanently" satisfied, it will not act again. Condition (i) helps ensure that the $\mathcal{N}$ strategy doesn't interfere too often with a $\mathcal{P}_j$ requirement.]

If $\mathcal{Q} = \mathcal{R}_{e,i,n}$: Let $m = m_{e,i,n,s-1}$, or $m = 0$ if $s = 0$. Check whether $n \in W_{e,s}^{A_{1-i,s}}$ by an α-believable computation.

Case 1: n is not in $W_{e,s}^{A_{1-i,s}}$ by an α-believable computation. Check whether $\langle n, m\rangle \in V_{e,i,s}^{A_{i,s}}$. If not, let $m_{e,i,n,s} = m$. Otherwise, let $m_{e,i,n,s} = m+1$ and add all $\{\langle n, m, \sigma\rangle : \sigma \in 2^{<\omega}\}$ to $V_{e,i}$. [This is to meet the part of the requirement involving $k < m_{e,i,n}$.]

Case 2: n is in $W_{e,s}^{A_{1-i,s}}$ by an α-believable computation. In this case, we leave $m_{e,i,n,s} = m$. Check whether $\langle n, m\rangle$ is already in $V_{e,i,s}^{A_{i,s}}$. If so, let $r_{\alpha,s}$ be either the use of $V_{e,i,s}^{A_{i,s}}(\langle n, m\rangle)$ or the use of $W_{e,s}^{A_{1-i,s}}(n)$, whichever is larger. If, on the other hand, $\langle n, m\rangle \notin V_{e,i,s}^{A_{i,s}}$, let σ be the shortest initial segment of $A_{i,s}$ satisfying:

i. $r_{<\alpha,s} < |\sigma|$;

ii. $|\sigma|$ is greater than the use of $W_{e,s}^{A_{1-i,s}}$; and

iii. for each proper initial segment $\beta \subsetneq \alpha$ assigned a requirement $\mathcal{P}_j$ with outcome $\alpha(|\beta|) = \mathtt{1}$, there exists an x of the form $x = \langle j, k\rangle$ for some k such that $r_{\beta,s} < x < |\sigma|$ and $\sigma(x) = 0$.

Add $\langle n, m, \sigma\rangle$ to $V_{e,i}$, and set $r_{\alpha,s}$ to equal $|\sigma|$. [The intuition behind condition (iii) is that the existence of such an x with $\sigma(x) = 0$ protects against the possibility that α's belief about the outcome of β – that $D^{[j]}$ is finite – might be wrong. If it is wrong, the computation based on σ will automatically be injured by the action of β or some other node assigned $\mathcal{P}_j$, and $\langle n, m\rangle$ will be removed automatically from $V_{e,i,s}^{A_{i,s}}$.]

1.2.4 Verification

Define the *true path* $\mathtt{tr} \in \{\mathtt{0}, \mathtt{1}\}^\omega$ as the leftmost path which is visited infinitely often. That is, for all n, the initial segment $\mathtt{tr}\upharpoonright n$ is the $\leq_{\text{lex}}$-least node of length n that is accessible infinitely often. We call $\mathtt{tr}(n)$ the *true outcome* of $\mathtt{tr}\upharpoonright n$. A node α is *on the true path* if α is an initial segment of $\mathtt{tr}$. If a node α is assigned a particular requirement $\mathcal{Q}$ at all but finitely many stages, we say that α is *eventually assigned* $\mathcal{Q}$ and write $\mathsf{ev}(\alpha) = \mathcal{Q}$. If there is no such $\mathcal{Q}$, we leave $\mathsf{ev}(\alpha)$ undefined. We begin with two straightforward lemmas. The first of these is, in fact, independent of the construction in §1.2.3.

Lemma 1.2.4. *If α is a node and $\alpha \leq_{\text{lex}} \mathsf{tr}$, then $\mathsf{ev}(\alpha)$ is defined.*

Proof. By induction on the length of α. Choose a stage s_0 large enough that for each $s \geq s_0$ we have $\alpha \leq_{\text{lex}} \delta_s$, and each strict initial segment of α is assigned the same requirement at stage s as at stage s_0. The assignment scheme in §1.2.2 gives the same requirement to α at each stage $s \geq s_0$. □

The second lemma relies on the construction in §1.2.3 only in that an $\mathcal{R}$ requirement always has $\mathsf{0}$ as its outcome.

Lemma 1.2.5. *The function $n \mapsto \mathsf{ev}(\mathsf{tr} \restriction n)$ is a bijection between ω and the set $\{\mathcal{P}_e, \mathcal{N}_{e,i}, \mathcal{R}_{e,i,n} : e, n \in \omega,\ i < 2\}$ of all requirements.*

Proof. For each n, let u_n be the number of $\ell < n$ for which $\mathsf{ev}(\mathsf{tr} \restriction \ell)$ is an $\mathcal{R}$ requirement, and let v_n be the total number of nodes $\beta \leq_{\text{lex}} \mathsf{tr} \restriction n$ that are ever accessible (which is finite by the definition of tr). From the assignment scheme in §1.2.2 we know that $\mathsf{ev}(\mathsf{tr} \restriction n)$ is an $\mathcal{R}$ requirement if $u_n < v_n/2$, and a $\mathcal{P}$ or $\mathcal{N}$ requirement if $u_n \geq v_n/2$.

It suffices to check that $\mathsf{ev}(\mathsf{tr} \restriction n)$ is an $\mathcal{R}$ requirement infinitely often, and a $\mathcal{P}$ or $\mathcal{N}$ requirement infinitely often. A few observations: (i) if $\mathsf{ev}(\mathsf{tr} \restriction n)$ is a $\mathcal{P}$ or $\mathcal{N}$ requirement, then $u_{n+1} = u_n$ and $v_{n+1} > v_n$; and (ii) if $\mathsf{ev}(\mathsf{tr} \restriction n)$ is an $\mathcal{R}$ requirement, then $u_{n+1} = 1 + u_n$ and $v_{n+1} = 1 + v_n$ (since the outcome must be $\mathsf{0}$, and $\beta <_{lex} \alpha^\frown 0$ implies either $\beta <_{lex} \alpha$ or $\beta = \alpha$). If cofinitely many $\mathsf{ev}(\mathsf{tr} \restriction n)$ were $\mathcal{P}$ or $\mathcal{N}$ requirements, then by definition $u_n \geq v_n/2$ cofinitely often, eventually contradicting (i); while if cofinitely many $\mathsf{ev}(\mathsf{tr} \restriction n)$ were $\mathcal{R}$ requirements, then by definition $u_n < v_n/2$ for cofinitely many n, eventually contradicting (ii). This completes the proof. □

Now we check that each requirement is met. We do this in two steps: first, in Propositions 1.2.6, we argue that nodes along the true path act infinitely often if and only if they are eventually assigned a $\mathcal{P}$ requirement with the infinitary $\mathsf{0}$ as their true outcome; and then in Proposition 1.2.8 we argue that these nodes' actions satisfy their respective requirements. For $\mathcal{P}$ and $\mathcal{N}$ requirements, the verification similar to the usual proof of the Sacks jump inversion; the method for $\mathcal{R}$ is new but straightforward. The remainder of this section makes full use of the construction in §1.2.3.

Proposition 1.2.6. *If $\alpha \leq_{\text{lex}} \mathsf{tr}$, then α acts infinitely often if and only if $\alpha \subseteq \mathsf{tr}$, $\mathsf{ev}(\alpha)$ is a $\mathcal{P}$ requirement, and its true outcome $\mathsf{tr}(|\alpha|)$ is $\mathsf{0}$.*

Proof. Since $\{\alpha : \alpha \leq_{\text{lex}} \mathsf{tr}\}$ is well-ordered by $\leq_{\text{lex}}$, we may work by induction on $\leq_{\text{lex}}$. Fix α. If α is strictly to the left of the true path, then the result is immediate, so assume that $\alpha = \mathsf{tr} \upharpoonright n$ for some n. Fix s_0 such that α is accessible at stage s_0, and large enough that every $\beta <_{\text{lex}} \alpha$ meets the inductive hypothesis before stage s_0 and $\alpha <_{\text{lex}} \delta_s$ for all $s \geq s_0$. In particular, the restraint $r_{<\alpha,s}$ is constant at stages $s \geq s_0$, and α is assigned the same requirement $\mathcal{Q} = \mathsf{ev}(\alpha)$ at all stages $s \geq s_0$. Consider the possible values of $\mathcal{Q}$.

Case 1: $\mathcal{Q} = \mathcal{P}_e$. By the definition of an action for a $\mathcal{P}$ requirement, α acts infinitely often if and only if it has the infinitary outcome $\mathsf{0}$ infinitely often, which happens if and only if its true outcome is $\mathsf{0}$.

Case 2: $\mathcal{Q} = \mathcal{N}_{e,i}$. If α does not act after stage s_0, there is nothing to prove; so let $s \geq s_0$ be least such that α acts, using the restraint $r_{\alpha,s}$ to preserve an α-believable computation $\Phi_{e,s}^{A_{1-i,s}}(x)\downarrow= y$ with $y \neq A_{i,s}(x)$. Because the computation is α-believable, and because, by choice of s_0, the only higher-priority nodes acting after stage s are initial segments $\beta \subseteq \alpha$ assigned a $\mathcal{P}_j$ requirement with true outcome $\mathsf{0}$, the computation $\Phi_{e,t}^{A_{1-i,t}}(x) = y$ continues to be α-believable as long as $r_{\alpha,t}$ does not decrease. Furthermore, since the $\mathcal{N}$-action of α stipulates as point (i) x is not of the form $\langle j, k\rangle$ for any such $\mathcal{P}_j$, the disagreement $y \neq A_{i,s}(x)$ is also preserved. Although α may act again after stage s to preserve some other computation with lesser use or lesser x, this happens at most finitely often, as x and the use are chosen to be minimal. Therefore α acts at most finitely many times.

Case 3: $\mathcal{Q} = \mathcal{R}_{e,i,n}$**.** We claim that $m_{e,i,n,s}$ is constant for $s > s_0$. Suppose for a contradiction that $s > s_0+1$ is the least stage at which some node assigned $\mathcal{R}_{e,i,n}$ acts by setting $m_{e,i,n,s} = m_{e,i,n,s-1} + 1$. This action is in response to $\langle n, m_{e,i,n,s-1}\rangle$ being in $V_{e,i,s}^{A_{i,s}}$ but n not being in $W_{e,s}^{A_{1-i,s}}$. Let β be the node that had placed the element in $V_{e,i}$ in response to a β-believable computation, and let γ be the node that had injured this computation by placing an element into A_{1-i} below the use. Then $\gamma \leq_{lex} \beta$, i.e. γ has higher priority, and γ acted after stage s_0, since otherwise α would already have dealt with this disagreement, or set up a restraint to prevent it, at stage s_0. Furthermore, γ does not extend α, or again α would have set up a restraint to prevent its action. Hence by choice of s_0, γ and β are strictly to the right of α. Since all initial segments of β which are not initial segments of α are assigned an $\mathcal{R}$ requirement (this is clear from the assignment scheme), and γ does not have an $\mathcal{R}$ (as it changes A_{1-i}), γ is strictly to the left of β, that is, they have a common initial segment δ with $\gamma(|\delta|) = \mathtt{0}$ and $\beta(|\delta|) = \mathtt{1}$. But then at the stage at which γ was accessible, δ was assigned a $\mathcal{P}$ requirement which acted by adding elements to A_i below the use of β's coding, and so (by condition (iii) in the $\mathcal{R}$-action of β) γ itself removed $\langle n, m_{e,i,n,s-1}\rangle$ from $V_{e,i}^{A_i}$ before γ had a chance to act. This is the desired contradiction. □

We are ready to begin checking that requirements are satisfied. We begin with the $\mathcal{P}$ requirements, as they will be useful in checking the others.

Lemma 1.2.7. *Every $\mathcal{P}$ requirement is satisfied.*

Proof. Fix a requirement $\mathcal{P}_e$, and let $\alpha \subseteq \mathsf{tr}$ be such that $\mathsf{ev}(\alpha) = \mathcal{P}_e$. Let s_0 be as in the proof of Proposition 1.2.6 and let $r = r_{<\alpha,s_0}$. If $D^{[e]} = \omega$ then there are infinitely many stages s at which new elements enter $D^{[e]}_s$, and so there are infinitely many stages at which α acts by adding elements to $A^{[e]}_0$ and $A^{[e]}_1$. In the limit, $A^{[e]}_0$ and $A^{[e]}_1$ contain all $k \geq r$, and so $A^{[e]}_0 =^* D^{[e]} =^* A^{[e]}_1$, as required.

If, on the other hand, $D^{[e]}$ is finite, it is easy to see that after some stage s no node assigned $\mathcal{P}_e$ ever again adds elements to $A^{[e]}_i$. We claim in addition that there is a stage s after which no node assigned an $\mathcal{N}$ requirement adds elements to $A^{[e]}_i$. By condition (i) of the $\mathcal{N}$ strategy, we need only consider nodes β of length $\leq e$. If β is strictly left of the true path, it eventually stops acting by definition of the true path; if β is strictly to the right of the true path, then eventually it is assigned an $\mathcal{R}$ requirement instead of an $\mathcal{N}$ requirement; and if β is on the true path, then by the previous Proposition, β either stops acting or is assigned a $\mathcal{P}$ requirement. □

Proposition 1.2.8. *Every requirement is satisfied.*

Proof. We have already dealt with the $\mathcal{P}$ requirements in the previous Lemma. Fix a requirement $\mathcal{Q}$ of the form $\mathcal{N}_{e,i}$ or $\mathcal{R}_{e,i,n}$, and let $\alpha \subseteq \mathsf{tr}$ such that $\mathsf{ev}(\alpha) = \mathcal{Q}$. Let s_0 be as in the proof of Proposition 1.2.6, and let $r = r_{<\alpha,s_0}$. Assume by induction that the requirements assigned to each proper initial segment of α are eventually satisfied. Of course, our methods depend on whether $\mathcal{Q}$ is an $\mathcal{N}$ or an $\mathcal{R}$ requirement.

Case 1: $\mathcal{Q} = \mathcal{N}_{e,i}$. Suppose for a contradiction that $\Phi_e^{A_{1-i}} = A_i$. Choose a j such that $D^{[j]}$ is finite, j is larger than the restraint r, and no $\beta \leq_{\text{lex}} \alpha$ is ever assigned $\mathcal{P}_j$. (Such a j exists because C is nonrecursive, and hence not cofinite.) Let $\alpha^* \subseteq \mathsf{tr}$ be such that $\mathsf{ev}(\alpha^*) = \mathcal{P}_j$, and notice that $\alpha \subsetneq \alpha^*$. As $A^{[j]}_i$ is finite by Lemma 1.2.7, there is an $x = \langle j, k\rangle$ such that $k \notin A^{[j]}_i$, so that $\Phi_e^{A_{1-i}}(x) = 0 = A_i(x)$. Since by Lemma 1.2.7 every $\mathcal{P}$ requirement assigned to an initial segment of α^* is satisfied, there is an $s \geq s_0$ with $\alpha \subseteq \delta_s$ such that $\Phi_{e,t}^{A_{1-i,t}}(x) = 0$ is α-believable for all $t \geq s$. But then α acts at or before this stage s and preserves a disagreement – a contradiction.

Case 2: $\mathcal{Q} = \mathcal{R}_{e,i,n}$. We saw in the proof of Proposition 1.2.6 that $m = m_{e,i,n,s}$ is constant when $s > s_0$. If $n \in W_e^{A_{1-i}}$, then, again appealing to the Lemma, $n \in W_{e,s}^{A_{1-i,s}}$ by an α-believable computation for large enough s, and so α eventually sets a restraint (and possibly adds to $V_{e,i}$) to preserve a computation $\langle n, m\rangle \in V_{e,i}^{A_i}$ as desired. If $n \notin W_e^{A_{1-i}}$, suppose for a contradiction that $\langle n, m\rangle \in V_{e,i}^{A_i}$. Then there is an $s > s_0$ such that $\langle n, m\rangle \in V_{e,i}^{A_i}$ and $n \notin W_e^{A_{1-i}}$. Since $\alpha \leq_{\text{lex}} \delta_s$ by choice of s_0, there is a $\beta \subseteq \delta_s$ assigned the requirement $\mathcal{R}_{e,i,n}$ at stage s [this is immediate from the assignment scheme and the fact that $\delta_s > |\alpha|$]. But then β should act at stage s by incrementing m, a contradiction. $\square$

It remains only to verify that $A_0', A_1' \leq_T C$. We use the following:

Lemma 1.2.9. *The true path* tr *is recursive in* C.

Proof. Using an oracle for C, we construct tr by recursion by building finite initial segments $\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \mathsf{tr}$. Begin with $\alpha_0 = \emptyset$, the empty string. For the recursive step, suppose we have defined $\alpha_n = \mathsf{tr} \restriction n$. Use $0'$ (which is recursive in C) to find out exactly how many s there are such that $\delta_s \leq_{\text{lex}} \alpha_n$, and hence to compute $\mathsf{ev}(\alpha_n)$. If $\mathsf{ev}(\alpha_n)$ is an $\mathcal{N}$ or $\mathcal{R}$ requirement, then the true outcome is $\mathsf{0}$, so we let $\alpha_{n+1} = \alpha_n{}^\frown\mathsf{0}$. On the other hand, if $\mathsf{ev}(\alpha_n)$ is $\mathcal{P}_e$, then by Proposition 1.2.8 the true outcome $\mathsf{tr}(n)$ is $\mathsf{0}$ if $D^{[e]}$ is infinite, and $\mathsf{1}$ otherwise. In other words, $\mathsf{tr}(n)$ is $\mathsf{0}$ if $e \notin C$, and $\mathsf{1}$ otherwise. Use the C oracle to define $\alpha_{n+1} = \alpha^\frown\mathsf{0}$ or $\alpha_{n+1} = \alpha^\frown\mathsf{1}$, as appropriate. $\square$

Proposition 1.2.10. $A_0', A_1' \leq_T C$.

Proof. We show that C can compute $\{e : e \in W_e^{A_i}\}$ for either value of i. Using the method of Lemma 1.2.9, use C to find, uniformly in e, a node α on the true path such that $\mathsf{ev}(\alpha) = \mathcal{R}_{e,i,e}$. As in the proof of Proposition 1.2.8, if $e \in W_e^{A_{1-i}}$, then eventually this computation is α-believable, so α acts (and succeeds) in preserving a coding $\langle e, m\rangle \in V_{e,i,s}^{A_{i,s}}$; on the other hand, if α acts at a stage after s_0 (defined as in the proof of Proposition 1.2.6) to preserve a such coding, then it also preserves $e \in W_e^{A_{1-i}}$. An oracle C can decide, using a query to $0'$, whether α acts in this way, and hence whether $e \in W_e^{A_{1-i}}$. $\square$

This completes the proof of Theorem 1.1.9.

1.3 Proving part (2) of the Theorem

Here we give a quick summary of the alterations needed to convert the proof of Theorem 1.1.9 part (1) into a proof of part (2). Fix a $\mathbf{c}$, C, and D as in the beginning of Section 1.3. Build two sets A_0, A_1 meeting the following requirements, for all e, i, n:

$\mathcal{P}_e : A_0^{[e]} =^* D^{[e]}$.

$\mathcal{N}_e : A_1 \neq \Phi_e^{A_0}$.

$\mathcal{R}_{e,n}$: There is an r.e. set V_e, which does not depend on n, and a number $m_{e,n}$ computable uniformly in n from $0'$ such that $n \in W_e^{A_0 \oplus A_1} \Leftrightarrow \langle n, m_{e,n}\rangle \in V_e^{A_0}$. Furthermore, $k < m_{e,n}$ implies $\langle n, k\rangle \in V_e^{A_0 \oplus A_1}$, and $k > m_{e,n}$ implies $\langle n, k\rangle \notin V_e^{A_0 \oplus A_1}$.

Then the required degrees are $\mathbf{b}_0 = \deg(A_0)$, and $\mathbf{b}_1 = \deg(A_0 \oplus A_1)$. The differences to keep in mind when adapting the construction are:

- $\mathcal{P}_e$ alters A_0 but never A_1.
- $\mathcal{N}_e$ alters A_1 but never A_0.

- $\mathcal{R}_{e,n}$ uses $A_0 \oplus A_1$ as an oracle instead of A_0 or A_1.
- The notion of α-believable computations $W_e^{A_0 \oplus A_1}(n)$ is adapted to allow for the fact that $\mathcal{P}$ requirements add elements to columns of A_0 but not to A_1.

From here the the proof proceeds by a sequence of lemmas analogous to that in Subsection 1.2.4.

1.4 From the Conjecture to Rigidity

In this section we give a proof based on Conjecture 1.1.5 of the rigidity of $\mathbf{R}$. If the Conjecture held then by Lemma 1.2.2 we would know that, for $\mathbf{a}, \mathbf{b} \in \mathbf{H}_2$, $\mathbf{a} = \mathbf{b} \Leftrightarrow \mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$: If $\mathbf{a} \neq \mathbf{b}$, then one would be not below the other and the Conjecture would supply a $\mathbf{c}$ in one of $\mathsf{JB}(\mathbf{a})$ or $\mathsf{JB}(\mathbf{b})$ but not the other.

Now fix any automorphism Φ of $\mathbf{R}$. By Nies, Shore and Slaman [52], $\mathbf{H}_2$ is definable in $\mathbf{R}$ and so if $\mathbf{a} \in \mathbf{H}_2$ then $\mathbf{\Phi}(\mathbf{a}) \in \mathbf{H}_2$. Now $\mathbf{H}_2$ is an automorphism basis for $\mathbf{R}$. (Indeed, by Lerman [41] every jump class is one but for $\mathbf{H}_2$ it follows easily from Theorem 1.1.2: If $\Phi(\mathbf{x}) = \mathbf{y} \neq \mathbf{x}$ then by Theorem 1.1.2 there is a $\mathbf{z} \in \mathbf{H}_2$ such that $\mathbf{z}$ is above one of $\mathbf{x}$ and $\mathbf{y}$ but not the other for the desired contradiction.) So to establish rigidity (based on our Conjecture) it would suffice to prove that $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\Phi(\mathbf{a}))$ for any $\mathbf{a} \in \mathbf{H}_2$. Assuming the Conjecture, we in fact show that JB is invariant under Φ, i.e. $\mathsf{JB}(\mathbf{x}) = \mathsf{JB}(\Phi(\mathbf{x}))$ for any $\mathbf{x} \in \mathbf{R}$.

We begin with the double jump version of JB: $\mathsf{DJB}(\mathbf{c}) = \{\mathbf{x}'' : \mathbf{x} \leq \mathbf{c}\}$. By using Lemma 1.2.2 both as stated and relativized, we see that for r.e. $\mathbf{c}$, $\mathsf{DJB}(\mathbf{c}) = \{\mathbf{x}'' : \mathbf{x} \leq \mathbf{c}\ \&\ \mathbf{x}$ is r.e.$\}$.

Claim 1. *If $\mathbf{x}, \mathbf{y} \in \mathbf{H}_2$ and $\mathsf{DJB}(\mathbf{x}) = \mathsf{DJB}(\mathbf{y})$, then $\mathbf{x}' = \mathbf{y}'$. Moreover, if $\mathbf{y} = \Phi(\mathbf{x})$ and $\mathbf{x} \in \mathbf{H}_2$ then then $\mathbf{x}' = \mathbf{y}'$.*

Proof. Suppose we have $\mathbf{x}$ and $\mathbf{y}$ as in the hypotheses of the Claim but $\mathbf{x}' \neq \mathbf{y}'$. Without loss of generality we may assume that $\mathbf{x}' \not\leq \mathbf{y}'$. As $\mathbf{x}', \mathbf{y}'$ are high (and so high$_2$) relative to $\mathbf{0}'$, we can apply the Conjecture relativized to $\mathbf{0}'$ to the degrees $\mathbf{x}' \not\leq \mathbf{y}'$. This gives us a $\mathbf{z}$ $REA(\mathbf{0}'')$ which is $REA(\mathbf{x}')$ but not $REA(\mathbf{y}')$. Next apply Theorem 1.1.2 relative to $\mathbf{x}$ with $\mathbf{0}'$ playing the role of $\mathbf{c}$ to get $\mathbf{f}$ $REA(\mathbf{x})$, $\mathbf{f} > \mathbf{0}'$ and $\mathbf{f}' = \mathbf{z}$. Finally, apply Theorem 1.1.2 with $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{f}$ playing the roles of $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{z}$, respectively, to get an r.e. $\mathbf{g} < \mathbf{x}$ with $\mathbf{g}' = \mathbf{f}$. Thus $\mathbf{g}'' = \mathbf{z}$ and so $\mathbf{z} \in \mathsf{DJB}(\mathbf{x})$. On the other hand, since $\mathbf{z}$ is not $REA(\mathbf{y}')$, $\mathbf{z} \notin \mathsf{DJB}(\mathbf{y})$ for the desired contradiction.

For the second part of the Claim, we note that by Nies, Shore and Slaman [52], not only is $\mathbf{y} \in \mathbf{H}_2$ definable in $\mathbf{R}$ but each of the double jump classes (i.e. the sets $\{\mathbf{c} \in \mathbf{R} | \mathbf{c}'' = \mathbf{d}\}$ for any $\mathbf{d}$ $REA(\mathbf{0}')$) are definable in $\mathbf{R}$. Thus $\mathsf{DJB}(\mathbf{c})$ is invariant under Φ and so $\mathsf{DJB}(\mathbf{x}) = \mathsf{DJB}(\mathbf{y})$ and we are done by the first part of the Claim. □

We next wish to prove that the jump is invariant, i.e. for any $\mathbf{x}, \mathbf{y}$ with $\Phi(\mathbf{x}) = \mathbf{y}$, $\mathbf{x}' = \mathbf{y}'$. We consider another operator on $\mathbf{R}$: $\mathsf{JA}(\mathbf{x}) = \{\mathbf{c}' | \mathbf{c} \geq \mathbf{x} \ \&\ \mathbf{c} \in \mathbf{H}_2\}$.

Claim 2. *For any* $\mathbf{x}, \mathbf{y} \in \mathbf{R}$, *if* $\mathsf{JA}(\mathbf{x}) = \mathsf{JA}(\mathbf{y})$, *then* $\mathbf{x}' = \mathbf{y}'$. *Moreover, for any* $\mathbf{x} \in \mathbf{R}$ *and* $\mathbf{y} = \Phi(\mathbf{x})$, $\mathbf{x}' = \mathbf{y}'$.

Proof. If the first assertion fails, assume, without loss of generality, that $\mathbf{y}' \not\leq \mathbf{x}'$. By Theorem 1.1.2 relative to $\mathbf{0}'$ with $\mathbf{x}'$, $\mathbf{0}''$, $\mathbf{y}'$, and $\mathbf{0}'''$ playing the roles of $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{z}$, respectively, we get an $\mathbf{f} > \mathbf{x}'$ with $\mathbf{f}' = \mathbf{0}'''$ which is $REA(\mathbf{0}')$ but not above $\mathbf{y}'$. By Theorem 1.1.2 again we have an r.e. $\mathbf{c} > \mathbf{x}$ with $\mathbf{c}' = \mathbf{f}$ and so $\mathbf{c} \in \mathbf{H}_2$. Thus $\mathbf{c}' = \mathbf{f} \in \mathsf{JA}(\mathbf{x})$. On the other hand, for every $\mathbf{v} \in \mathsf{JA}(\mathbf{y})$, $\mathbf{v} \geq \mathbf{y}'$ but $\mathbf{f}$ is not above $\mathbf{y}'$ for the desired contradiction.

For the second part of the Claim, note that by the previous Claim, $\mathsf{JA}(\mathbf{x})$ is invariant under Φ: If $\mathbf{z} \in \mathsf{JA}(\mathbf{x})$ then $\mathbf{z} = \mathbf{c}'$ for some $\mathbf{c} \in \mathbf{H}_2$ with $\mathbf{c} \geq \mathbf{x}$. By the previous Claim, $\mathbf{c}' = \Phi(\mathbf{c})'$ (and so, in particular $\Phi(\mathbf{c}) \in \mathbf{H}_2$). As $\Phi(\mathbf{c}) \geq \Phi(\mathbf{x})$, $\mathbf{z} = \mathbf{c}' = \Phi(\mathbf{c})' \in \mathsf{JA}(\Phi(\mathbf{x}))$. Similarly, if $\mathbf{z} \in \mathsf{JA}(\Phi(\mathbf{x}))$ then $\mathbf{z} = \mathbf{c}'$ for some $\mathbf{c} \in \mathbf{H}_2$ with $\mathbf{c} \geq \Phi(\mathbf{x})$. There is then a $\mathbf{y}$ with $\Phi(\mathbf{y}) = \mathbf{c}$ and so $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \in \mathbf{H}_2$. Again, by the previous Claim, $\mathbf{y}' = \Phi(\mathbf{y})' = \mathbf{c}' = \mathbf{z}$ and so $\mathbf{z} \in \mathsf{JA}(\mathbf{x})$ as desired. $\square$

We can now complete our proof of the rigidity of $\mathbf{R}$ from the conjecture by noting that by this last Claim and the definition of JB, JB is invariant under Φ: If $\mathbf{c} \leq \mathbf{x}$ then $\Phi(\mathbf{c}) \leq \Phi(\mathbf{x})$ and by the last Claim $\mathbf{c}' = \Phi(\mathbf{c})'$. Thus $\mathsf{JB}(\mathbf{x}) \subseteq \mathsf{JB}(\Phi(\mathbf{x}))$. Similarly, if $\mathbf{d} \leq \Phi(\mathbf{x})$ then $\mathbf{d} = \Phi(\mathbf{c})$ for some $\mathbf{c} \leq \mathbf{x}$ and so $\mathbf{c}' = \mathbf{d}'$ and $\mathsf{JB}(\Phi(\mathbf{x})) \subseteq \mathsf{JB}(\mathbf{x})$ giving rigidity as in the second paragraph of this section.

1.5 Questions and Observations

In this section, we pose a number of questions that naturally extend Proposition 1.1.6, Corollary 1.1.8 and Theorem 1.1.9 and make some observations which impede or even restrict such possibilities.

The first set of questions deal with the issue of when r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ in the same jump class, given say by $\mathbf{c} \in \mathbf{REA}(\mathbf{0}')$, have $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ by extending Corollary 1.1.8.

Question 1.5.1. When (for $\mathbf{c} \in \mathbf{REA}(\mathbf{0}')$ and $\mathbf{c}'' \neq \mathbf{0}'''$) do we have r.e. $\mathbf{a} \neq \mathbf{b}$ with jump $\mathbf{c}$ such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$? Of course, we must here at least have $\mathbf{c} > \mathbf{0}'$.

The noninversion theorem of Shore [62] gives some examples of degrees distinguished by the JB operator along these lines beyond those given by Corollary 1.1.8. Indeed, it supplies two upward cones in $\mathbf{R}$ such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ for any incomplete $\mathbf{a}$ and $\mathbf{b}$ in each cone and a cone of jump classes all realized by degrees in each of these two cones.

Proposition 1.5.2. *There are incomparable r.e.* $\mathbf{c}$ *and* $\mathbf{d}$ *such that for* $\mathbf{0}' > \mathbf{a} \geq \mathbf{c}$ *and* $\mathbf{0}' > \mathbf{b} \geq \mathbf{d}$, $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$. *Moreover, there is a* $\mathbf{w} \in \mathbf{REA}(\mathbf{0}')$ *with* $\mathbf{w} < \mathbf{0}''$ *such that for any* $\mathbf{z} \geq \mathbf{w}$ *with* $\mathbf{z} \in \mathbf{REA}(\mathbf{0}')$, *there are* $\mathbf{a} \geq \mathbf{c}$ *and* $\mathbf{b} \geq \mathbf{d}$ *with* $\mathbf{a}' = \mathbf{z} = \mathbf{b}'$.

Proof. By Shore [62, Theorem 1.1], there are $\mathbf{a}_0, \mathbf{a}_1 \in \mathbf{REA}(\mathbf{0}')$ such that $\mathbf{a}_0 \vee \mathbf{a}_1 < \mathbf{0}'$ and if $\mathbf{u} < \mathbf{0}'$ then not both $\mathbf{a}_0$ and $\mathbf{a}_1$ are $REA(\mathbf{u})$. Now take r.e. $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{c}' = \mathbf{a}_0$ and $\mathbf{d}' = \mathbf{a}_1$. Consider now any incomplete $\mathbf{a} \geq \mathbf{c}$ and $\mathbf{b} \geq \mathbf{d}$. It is clear that $\mathbf{a}_0 \in \mathsf{JB}(\mathbf{a})$ and $\mathbf{a}_1 \in \mathsf{JB}(\mathbf{b})$. On the other hand, if $\mathbf{a}_1 \in \mathsf{JB}(\mathbf{a})$ then $\mathbf{a}$ would be complete and so $\mathbf{a}_1 \notin \mathsf{JB}(\mathbf{a})$. Similarly $\mathbf{a}_0 \notin \mathsf{JB}(\mathbf{b})$ as required. Of course, $\mathbf{c}$ and $\mathbf{d}$ are incomparable as otherwise the larger of the two would be an incomplete $\mathbf{u}$ in which both $\mathbf{a}_0$ and $\mathbf{a}_1$ would be r.e.

Finally, by Theorem 1.1.2, we may take $\mathbf{a}_0 \vee \mathbf{a}_1$ as the $\mathbf{w}$ required in the Proposition. □

All the examples of pairs of incomplete r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ that we have seen provide, as far as we have specifically determined, only incomparable $\mathbf{a}$ and $\mathbf{b}$. As we know of no others, we ask for comparable such pairs.

Question 1.5.3. When (for $\mathbf{c} \in \mathbf{REA}(\mathbf{0}')$) do we have r.e. $\mathbf{a} < \mathbf{b} < \mathbf{0}'$ with $\mathbf{a}' = \mathbf{b}' = \mathbf{c}$ such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$?

Of course, we must here also at least have $\mathbf{c} > \mathbf{0}'$ but Proposition 1.1.6 and Corollary 1.1.8 do not supply an answer even for $\mathbf{c} = \mathbf{0}''$ or $\mathbf{c}' = \mathbf{0}'''$. We can ask for even more along the lines of distinguishing r.e. degrees.

Question 1.5.4. Is there, for every nonlow r.e. $\mathbf{a}$, an r.e. $\mathbf{b} \neq \mathbf{a}$ with $\mathbf{a}' = \mathbf{b}'$ and $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$? For which such $\mathbf{a}$ can we, in addition, choose $\mathbf{b}$ so that we have $\mathbf{a} < \mathbf{b}$, $\mathbf{b} < \mathbf{a}$ or $\mathbf{a} | \mathbf{b}$?

Using a result of Arslanov, Lempp and Shore [1], we can show that the strongest possible version of such a statement does not hold.

Proposition 1.5.5. *There is a nonlow r.e.* $\mathbf{c}$ *such that every incomplete r.e.* $\mathbf{b} \geq \mathbf{c}$ *with* $\mathbf{c}' = \mathbf{b}'$ *has* $\mathsf{JB}(\mathbf{c}) = \mathsf{JB}(\mathbf{b})$.

Proof. Arslanov, Lempp and Shore [1, Theorem 3.1] states that there is an incomplete nonrecursive r.e. A such that every set $REA(A)$ and recursive in $\mathbf{0}'$ is of r.e. degree. By the uniformity inherent in the proof of this result, we can apply the pseudojump inversion theorem of Jockusch and Shore [31] to get an r.e. C such that some set $A \in \mathbf{0}'$ has this property relative to C. That is, $C <_T A < C'$ and every set $REA(A)$ and recursive in C' is of degree r.e. in C. Thus if the degree $\mathbf{x} \in \mathbf{REA}(\mathbf{0}')$ (which, of course, is the same as $\mathbf{REA}(\mathbf{a})$) and $\mathbf{x} \leq \mathbf{c}'$ then some $X \in \mathbf{x}$ is $REA(A)$ and so $\mathbf{x} \in \mathbf{REA}(\mathbf{c})$. Thus, in particular, if $\mathbf{b} \geq \mathbf{c}$ and $\mathbf{b}' = \mathbf{c}'$ then then $\mathsf{JB}(\mathbf{c}) = \mathsf{JB}(\mathbf{b})$. □

We note that, as Arslanov, Lempp and Shore point out, the A they construct cannot be either low$_2$ or high. This translates into fact that the $\mathbf{c}$ produced in the above Proposition is neither high$_2$ nor low$_2$. We do not know any more about the possibilities for such $\mathbf{c}$ but there are several tempting possibilities. For example, could such a $\mathbf{c}$ also be least in its jump class with this maximal value of $\mathsf{JB}(\mathbf{c})$, i.e. could it be that for $\mathbf{x} \not\geq \mathbf{c}$, $\mathsf{JB}(\mathbf{x}) \neq \mathsf{JB}(\mathbf{c})$? If so, this would in a different way characterize $\mathbf{c}$ in terms of the JB operator. If not then, perhaps it might be minimal, i.e. for $\mathbf{x} < \mathbf{c}$, $\mathsf{JB}(\mathbf{x}) \subsetneq \mathsf{JB}(\mathbf{c})$ and so one would have characterized at least an antichain of degrees as the ones with this value of JB.

Moving in the other direction, i.e. towards stronger versions of Theorem 1.1.9 and Question 1.1.11, we can ask the following:

Question 1.5.6. Is there, for every nonrecursive, incomplete r.e. $\mathbf{a}$ an r.e. $\mathbf{b}$ with $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$ for which we can also guarantee that $\mathbf{b} > \mathbf{a}$, $\mathbf{b} < \mathbf{a}$ or $\mathbf{b}|\mathbf{a}$?

Note that as we pointed out after Definition 1.1.3, Theorem 1.1.1 shows that even to get a $\mathbf{b} < \mathbf{a}$ with $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$ we must assume that $\mathbf{a}$ is incomplete. It is hard to see how the assumption of the incompleteness of $\mathbf{a}$ can be used in the construction of a $\mathbf{b} < \mathbf{a}$.

We conclude with some methodological remarks about the construction that highlights a reverse mathematical issue. First, we note that despite the unusual type of argument about the assignment of requirements along the true path, the construction is still, by the usual criterion, a $0''$ one: In particular, our proof shows that $0''$ can compute the true path. Once one knows the fact that each requirement is eventually assigned to a fixed node along the true path, $0''$ can compute where and when this happens and so the precise way in which each requirement is satisfied.

Now, it is generally the case that $0''$ constructions can be carried out in $I\Sigma_2$. The anomaly here is that the proof that each requirement is eventually assigned to a fixed node along the true path (Lemma 1.2.5) and so of the fact that $0''$ can calculate all the outcomes of the construction seems to require $I\Sigma_3$. The point here is that in order to prove Lemma 1.2.5, we use an instance of the principle that any finite iterate $f_m = \underbrace{f \circ \cdots \circ f}_{m \text{ times}}$ of a total Π_2-definable function f is itself a total function. (In our case, $f(n)$ is the number of elements which are ever accessible, and which are $\leq_{\text{lex}} \mathsf{tr} \restriction n$, where tr denotes the true path. This f comes from the scheme for assigning requirements to nodes, which can be found in §1.2.) This principle is known as Π_2 *recursion*. It is denoted by $T\Pi_2$ in Hajek and Pudlak [23] in the setting of first order arithmetic and PREC_3 in Hirschfeldt and Shore [29] in the setting of second order arithmetic. In each setting, it is shown equivalent to $I\Sigma_3$. Thus our proof, unlike previous examples, uses $I\Sigma_3$ and so more induction than one would expect.

The solution to this problem is to use Shore blocking to assign blocks of requirements along the paths of the construction (and so along the true path). Thus, for example, instead of assigning at stage s a single requirement of some type ($\mathcal{R}_{e,i,n}$, $\mathcal{P}_e$ or $\mathcal{N}_{e,i}$) at a node α as in our construction, one assigns the block of the next requirements of the same type of size s (i.e. ones not yet on the path of the form $\mathcal{R}_{k,j,l}$ for $k, j, l < s$, $\mathcal{P}_j$ for $j < s$ or $\mathcal{N}_{j,i}$ for $j < s$, respectively). We do not know of another $0''$ construction that requires blocking along the paths of the priority tree to carry out the argument that the requirements are satisfied in $I\Sigma_2$.

CHAPTER 2

REVERSE MATHEMATICS OF MODEL THEORY

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2.1 Introduction

Simpson [63, Ch. II.8 and IV.3] laid the foundation for the study of first-order logic from the point of view of reverse mathematics. There he provided suitable definitions of objects such as theories and models in the language of second-order arithmetic, and proved versions of several important theorems, including the Soundness and Completeness Theorems, in the weak axiom system $\mathsf{RCA_0}$. In [63, Ch. IX.4] he began the study of model theory proper by formalizing and proving the existence theorem for recursively saturated models in the system $\mathsf{WKL_0}$. This work was motivated, however, by its applications to metamathematical conservation theorems. Recently, there has been a surge interest in the reverse mathematics of model theory *per se*, and researchers such as Harris, Hirschfeldt, Lange, Shore, and Slaman have undertaken a systematic study using Simpson's framework.

While much of this work has fallen into the familiar pattern of placing lists of theorems in correspondence with one of several known axiom systems—most often one of the Big Five isolated by Friedman [17, 18]—it has also enriched the field by suggesting totally new axiom systems. For example, Hirschfeldt, Shore, and Slaman [30], in studying the classical existence theorem for atomic models, isolated the new reverse-mathematical principles AMT and $\Pi^0_1\mathsf{G}$. Hirschfeldt, Lange, and Shore [28], drawing on work in effective model theory by Goncharov [21] and Peretyat′kin [54], have studied various versions of the classical existence theorem for homogeneous models, finding further connections with AMT and with induction principles such as $\mathsf{B}\Sigma^0_2$ and $\mathsf{I}\Sigma^0_2$, and discovering a new hierarchy of principles $\Pi^0_n\mathsf{GA}$ between $\mathsf{I}\Sigma^0_\mathsf{n}$ and $\mathsf{I}\Sigma^0_{\mathsf{n+1}}$ but incomparable with $\mathsf{B}\Sigma^0_{\mathsf{n+1}}$.

Given the known connections between reverse and effective mathematics (as described in, for example, Friedman, Simpson, and Smith [19]), it should come as no surprise that the reverse-mathematical approach to model theory also has strong connections with effective model theory. On the one hand, many known results and techniques from the effective setting can be formalized in RCA_0. On the other, the fact that many other results *cannot* be formalized in RCA_0 suggests new questions in effective mathematics.

It has typically been the case in effective model theory that when a particular object is being studied its complexity is tightly controlled, while that of other objects varies freely. An example that comes up frequently is the isomorphism relation: two models are *isomorphic* if there is an isomorphism between them. The Turing degree of the isomorphism is not normally considered, unless it is the main object of interest, as in the study of recursive stability or relative categoricity. Because it is unnatural in reverse mathematics to treat a model or theory differently from an isomorphism—all second-order objects obey the same basic set-existence axioms—our approach here must be more uniform. When interpreted in ω-models, our results over RCA_0 can be viewed as correspondingly uniform results in effective mathematics.

In this paper we address, within various subsystems of second-order arithmetic, the following two questions of basic model theory.

Q1. *Under what conditions is a complete theory $\aleph_0$-categorical?*

Q2. *For what finite values n may we have a complete theory with exactly n models up to isomorphism?*

We assume familiarity with reverse mathematics and with model theory. Subsections §2.1.1 and §2.1.2 describe some of our less standard notation, and provide a few useful lemmas in reverse mathematics and in model theory, respectively. Subsections §2.2.1 and §2.2.2 summarise our answers to the questions Q1 and Q2, respectively. Most of the proofs are deferred to the remainder of the paper, namely §§2.3–2.7. Each section among §§2.3–2.7 is built around a particular construction or technique, and is split into four parts: first, a brief description of the construction and its goals; second, a subsection giving the construction itself; third, a 'verification' subsection where basic properties are checked (such as completeness and consistency of a particular theory); and, finally, an 'applications' subsection where the construction is used to prove claims from §2.2.1 and §2.2.2.

Suitable machinery is introduced and developed as needed, including a WKL_0 version of the Henkin model construction in §2.5 and an RCA_0 version of the Fraïssé limit construction in §2.6. Unless otherwise stated, all reasoning is in RCA_0. A theorem's statement may be tagged with the axiom system in which it is being proved, such as RCA_0, ACA_0, or 'Classical' when reasoning in ZFC.

2.1.1 Notation for reverse mathematics

Most of our reverse-mathematical notation follows Simpson [63]. We use M and to denote the first- and second-order parts, respectively, of a model $\mathcal{M} = (M, , +_M, \cdot_M, 0_M, 1_M, <_M)$ of RCA_0. We typically assume, without mention, that we are working inside such a model; when we do mention the model we omit the operation symbols, writing simply $(M,)$. We say that a set $X \in$ is *finite* if it has an upper bound in M. We use the symbol $\{0,1\}^{<M}$ or $2^{<M}$ to denote the set of all finite binary strings in . We use $\mathsf{I}\Sigma^0_1$ to denote the axiom scheme of induction for Σ^0_1 formulas with parameters from M and . We also use the following notation.

Definition 2.1.1. Fix a set $Z \in$ in a model $(M,)$ of RCA_0.

(i) Given a sequence of sets $X_0, \ldots, X_{n-1} \in$, where $n \in M$ may be nonstandard, we define the *coded tuple* $\langle X_0, \ldots, X_{n-1} \rangle$ as the predicate:

$$\langle X_0, \ldots, X_{n-1} \rangle(\langle i, k \rangle) \iff k \in X_i.$$

Given a sequence of sets $X_0, X_1, \ldots \in$ with indices ranging over all of M, we define the *coded sequence* $\langle X_0, X_1, \ldots \rangle$ similarly:

$$\langle X_0, X_1, \ldots \rangle(\langle i, k \rangle) \iff k \in X_i.$$

We sometimes treat coded tuples and coded sequences as sets, for example by writing $\langle i, k \rangle \in \langle X_0, X_1, \ldots \rangle$. Depending on how the sets X_i are presented, a coded tuple or coded sequence may or may not to be an element of . In this paper, we usually point out when it is.

(ii) Given a set $Z \in$ and a number $s \in M$, let $K_s^Z = \{e < s : \Phi_{e,s}^Z(e) \text{ converges}\}$, where Φ_e is the e-th Turing functional. The *Turing jump enumeration for* Z is the coded sequence $\langle K_0^Z, K_1^Z, \ldots \rangle$. Note that the Turing jump enumeration exists in by Δ_1^0 comprehension. We let $K_{at\ s}^Z$ denote the set difference $K_s^Z - K_{s-1}^Z$.

(iii) The *Turing jump of* Z, written K^Z, is the Σ_1^0 predicate

$$K^Z(n) \iff (\exists s)[n \in K_s^Z].$$

We often write $n \in K^Z$ to mean $K^Z(n)$.

The following lemma shows how the Turing jump fits into reverse mathematics.

Lemma 2.1.2 (RCA_0)**.** *Let* $(M,)$ *be a model of* RCA_0*. Then* $(M,)$ *is a model of* ACA_0 *if and only if* K^Z *is an element of for every* $Z \in$*.*

Proof. See Simpson [63, Ex. VIII.1.12]. □

Lemma 2.1.2 allows us to obtain reversals from a principle P to ACA_0 by coding $\langle K_0^Z, K_1^Z, \ldots \rangle$ into an object and arguing that, if P holds, then we can use Δ_1^0 comprehension to recover K^Z. We use this method frequently, for example, in the proofs of Proposition 2.4.5 and Proposition 2.6.11.

2.1.2 Background and notation for model theory

All definitions are in the language of second-order arithmetic. Our definitions for basic model-theoretic terms such as *language*, *formula*, *sentence*, *structure*, *model*, *consistent*, and *satisfiable* are mostly as given in Simpson [63, Ch. II.8] and in Hirschfeldt, Lange, and Shore [28]. All structures have countably infinite domain unless otherwise specified. Given a language L, an *L-theory* is any set of L-sentences. A *complete L-theory* is a theory containing either ϕ or $\neg\phi$ for every L-sentence ϕ. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if there is an isomorphism between them. When we are working in a model $(M,)$ of RCA_0, the isomorphism must be an element of . A theory is $\aleph_0$*-categorical* if all of its models are isomorphic.

We shall need the following theorem.

Theorem 2.1.3 (RCA_0. Weak Completeness Theorem)**.** *Every deductively-closed consistent theory is satisfiable. In particular, every complete consistent theory is satisfiable, and every deductively-closed consistent theory can be extended to a complete consistent theory.*

Originally due to Gödel, the Weak Completeness Theorem 2.1.3 was formalized in effective mathematics by Morley and translated to reverse mathematics by Simpson [63, Thm II.8.4]. *Weak* is in the name to contrast this with the stronger statement, not provable in RCA_0, which does not include *deductively-closed* as a hypothesis:

Theorem 2.1.4. *The statement, 'Every consistent theory is satisfiable' is equivalent to* WKL_0 *over* RCA_0*.*

Proof. See Simpson [63, Thm IV.3.3]. □

One of the Weak Completeness Theorem's immediate consequences is the following theorem of Łos and Vaught.

Theorem 2.1.5. *(i) (Classical. Łos, Vaught.) If T is an L-theory with only one countable model, then for every L-sentence ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$.*

(ii) (RCA_0.) *Every deductively-closed theory with exactly one model up to isomorphism is complete.*

(iii) The statement of part (i) is equivalent to WKL_0 *over* RCA_0*.*

Proof. A proof of part (i) can be found in standard texts such as Marker [43]. Part (ii) and the forward direction of part (iii) are implicit in the proof given in Simpson [63, Ch. II.8] of the Weak Completeness Theorem 2.1.3.

For the reverse direction of (iii), assume that $\neg\mathsf{WKL}_0$ holds. By Theorem 2.1.4, there is a language L_0 and a consistent L_0-theory T_0 with no models. We may assume L_0 is a relational language. Let $L_1 = \{\leq\}$ be the language of partial orders, and let T_1 be the theory of dense linear orders without endpoints, which is $\aleph_0$-categorical in RCA_0. Define a new language $L = L_0 \cup L_1 \cup \{R\}$, where R is a new 0-ary relation, and an L-theory T by:

$$
\begin{aligned}
T = \{\neg R \to \phi : \phi \in T_0\} &\cup \{\neg R \to \text{all relations in } L_1 \text{ are empty}\} \\
\cup\{R \to \phi : \phi \in T_1\} &\cup \{R \to \text{all relations in } L_0 \text{ are empty}\}
\end{aligned}
$$

This T has exactly one model, but neither proves nor refutes the sentence R. □

Thus, in the system WKL_0, if we wish to show that a theory is complete, it is enough to construct a model and show that it is unique up to isomorphism. This is, in general, not enough in the weaker system RCA_0. Instead, we use a suitably effective notion of quantifier elimination.

Definition 2.1.6. (i) We say a theory T *has quantifier elimination* if, for every L-formula $\phi(\bar{x})$, there is a quantifier-free L-formula $\psi(\bar{x})$—possibly one of the formal logical symbols Tr or Fa—such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

(ii) We say a theory T *has effective quantifier elimination* if there is a function which takes as input any L-formula $\phi(\bar{x})$ and returns an L-formula $\psi(\bar{x})$—possibly Tr or Fa—such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

Any theory with effective quantifier elimination has quantifier elimination, and, in a relational language, any theory with quantifier elimination is complete. The following lemma, used in the work of Hirschfeldt, Shore, and Slaman [30], is our main tool for proving completeness of a theory.

Lemma 2.1.7 (RCA_0)**.** *Suppose T is a theory and there is a function which takes as input an L-formula $\theta(\bar{x}, y)$ which is a conjunction of literals and returns a quantifier-free L-formula $\psi(\bar{x})$ such that $T \vdash (\exists y)\theta(\bar{x}, y) \leftrightarrow \psi(\bar{x})$. Then T has effective quantifier elimination.*

Proof. Suppose such a function f exists, and fix any L-formula $\phi(\bar{x})$. We show how to produce a ψ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Suppose first that $\phi(\bar{x})$ is of the form $(\exists y)\theta(\bar{x}, y)$, where θ is quantifier-free. The usual proof of De Morgan's laws may be carried out in RCA_0, so we may assume that θ is in disjunctive normal form, say $\theta_0(\bar{x}, y) \vee \cdots \vee \theta_{n-1}(\bar{x}, y)$. Since RCA_0 is also strong enough to prove the distributivity of $\exists$ over $\vee$, we have $T \vdash \phi(\bar{x}) \leftrightarrow (\exists y)\theta_0(\bar{x}, y) \vee \cdots \vee (\exists y)\theta_{n-1}(\bar{x}, y)$. We may now use the provided function f to to find quantifier-free formulas $\psi_0(\bar{x}), \ldots, \psi_{n-1}(\bar{x})$ such that $T \vdash (\exists y)\theta_i(\bar{x}, y) \leftrightarrow \psi_i(\bar{x})$ for all $i < n$. Then $T \vdash \phi(\bar{x}) \leftrightarrow \psi_0(\bar{x}) \vee \cdots \vee \psi_{n-1}(\bar{x})$, so $\psi_0 \vee \cdots \vee \psi_{n-1}$ is the desired ψ.

Now suppose that $\phi(\bar{x})$ is a formula of arbitrary quantifier depth $n > 0$. Using the above procedure on the deepest quantifiers of ϕ, we can find a formula which is provably equivalent to ϕ and has quantifier depth $n-1$. Iterate this procedure using Δ^0_1 recursion to get a quantifier-free ψ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. □

The following definitions are of central importance to the study of $\aleph_0$-categorical theories.

Definition 2.1.8. Fix a natural number n, a language L, and a complete, consistent L-theory T.

(i) An *n-type of T* is a set $p(x_0, \ldots, x_{n-1})$ of formulas in variables taken from $\{x_0, \ldots, x_{n-1}\}$ such that $T \subseteq p(x_0, \ldots, x_{n-1})$ and, if $c_0, \ldots, c_{n-1}$ are new constants not in L, then the set

$$\{\phi(c_{i_0}, \ldots, c_{i_{k-1}}) : \phi(x_{i_0}, \ldots, x_{i_{k-1}}) \in p(x_0, \ldots, x_{n-1})\}$$

is a complete, consistent $L \cup \{c_0, \ldots, c_n\}$-theory. We sometimes abbreviate $p(x_0, \ldots, x_{n-1})$ to $p(\bar{x})$, or just p. We often omit n and call $p(\bar{x})$ simply a *type*.

(ii) A type $p(\bar{x})$ of T is *principal* if there is a formula $\phi(\bar{x}) \in p(\bar{x})$ such that $T \vdash \phi(\bar{x}) \rightarrow \psi(\bar{x})$ for all $\psi(\bar{x}) \in p(\bar{x})$. Otherwise, $p(\bar{x})$ is *nonprincipal*.

(iii) Suppose that $\mathcal{A}$ is a model of T and $p(\bar{x})$ is a type. We say that $\mathcal{A}$ *realizes* $p(\bar{x})$ if there is a tuple $\bar{a}$ from its domain such that $\mathcal{A} \models \phi(\bar{a})$ for every $\phi(\bar{x}) \in p(\bar{x})$. Otherwise, we say that $\mathcal{A}$ *omits* $p(\bar{x})$.

An RCA_0 version of the classical Type Omitting Theorem can be proved by an easy Henkin-style construction.

Theorem 2.1.9 (Classical and RCA_0. Type Omitting Theorem)**.** *Let T be a complete theory and $p(\bar{x})$ a nonprincipal type. There is a model of T that omits $p(\bar{x})$.*

Proof. See Harizanov [24, Theorem 6.1]. □

Much more intricate type-omitting theorems can be found in the work of Millar [49] in effective mathematics. Some of these have been studied in reverse mathematics by Hirschfeldt, Shore, and Slaman [30].

2.2 Main results

The main results of this paper fall into two classes, listed separately in §2.2.1 and §2.2.2. Section §2.2.1 deals with a theorem of Ryll-Nardzewski, Engeler, and Svenonius about $\aleph_0$-categorical theories and their n-types. Section §2.2.2 deals with theorems about theories, not necessarily $\aleph_0$-categorical, that have only finitely many models.[1]

2.2.1 Reverse mathematics and $\aleph_0$-categorical theories

Recall our first question:

Q1. *Under what conditions is a complete theory T $\aleph_0$-categorical?*

In the classical setting, Engeler [14], Ryll-Nardzewski [57], and Svenonius [69] independently discovered a number of properties characterising $\aleph_0$-categorical theories. Many such properties are now known. We focus on the following five:

Theorem 2.2.1 (Classical. Engeler; Ryll-Nardzewski; Svenonius)**.** *Let T be a complete, consistent theory, and let M denote the true natural numbers ω. The following are equivalent:*

(S1) *There is a function $f : M \to M$ such that, for all $n \in M$, T has exactly $f(n)$ distinct n-types.*

[1] These are sometimes called *Ehrenfeucht theories*.

(S2) *There is a function $f : M \to M$ such that, for all $n \in M$, T has no more than $f(n)$ distinct n-types.*

(S3) *T has only finitely many n-types, for each $n \in M$.*

(S4) *T is $\aleph_0$-categorical.*

(S5) *All types of T are principal.*

Our approach to the question Q1 is to explore the reverse-mathematical strength of Theorem 2.2.1, allowing nonstandard M. In other words, we replace Q1 with the more specific question:

Q1′. *What is the strength over RCA_0 of each implication $(\mathsf{S}i \to \mathsf{S}j)$?*

It is simple to check that the classical proofs of equivalence for principles (S1)–(S5), as found in standard texts such as Marker [43], all work in ACA_0. Over RCA_0, each implication therefore lies somewhere between RCA_0 and ACA_0.

The following table summarizes our results. Each implication $(\mathsf{S}i \to \mathsf{S}j)$ is equivalent to the principle named in the cell in row $(\mathsf{S}i)$ and column $(\mathsf{S}j)$; tautologies of the form $(\mathsf{S}i \to \mathsf{S}i)$ are greyed out; and any other blank cell means 'unknown'. Each of these equivalences is justified in one of Theorem 2.2.2, Theorem 2.2.3, and Theorem 2.2.4 below.

	(S1)	(S2)	(S3)	(S4)	(S5)
(S1)		RCA_0	RCA_0	RCA_0	RCA_0
(S2)	ACA_0		RCA_0	ACA_0	RCA_0
(S3)	ACA_0	ACA_0		ACA_0	RCA_0
(S4)			WKL_0		RCA_0
(S5)	ACA_0	ACA_0	ACA_0	ACA_0	

Table 2.1: Implications for Theorems 2.2.2 through 2.2.4.

We begin by isolating, in Theorem 2.2.2, the implications that require a detailed proof, indicating in each case where in this paper the proof can be found. We then list, in Theorem 2.2.3, several implications that are easily provable in RCA_0, giving in each case a short argument or reference. All other implications in the table follow by composing implications from Theorems 2.2.2 and 2.2.3, as outlined in the proof of Theorem 2.2.4.

Theorem 2.2.2. *(i)* $\mathsf{RCA}_0 \vdash (\mathit{S2} \to \mathit{S1}) \to \mathsf{ACA}_0$. *(Proposition 2.6.11)*

(ii) $\mathsf{RCA}_0 \vdash (\mathit{S2} \to \mathit{S4}) \to \mathsf{ACA}_0$. *(Corollary 2.6.14)*

(iii) $\mathsf{RCA}_0 \vdash (\mathit{S3} \to \mathit{S2}) \to \mathsf{ACA}_0$. *(Proposition 2.6.12)*

(iv) $\mathsf{RCA}_0 \vdash (\mathit{S5} \to \mathit{S3}) \to \mathsf{ACA}_0$. *(Proposition 2.4.5)*

(v) $\mathsf{RCA}_0 \vdash (\mathit{S4} \to \mathit{S3}) \leftrightarrow \mathsf{WKL}_0$. *(Propositions 2.3.5 and 2.5.6)*

(vi) $\mathsf{RCA}_0 \vdash (\mathit{S5} \to \mathit{S4}) \to \mathsf{ACA}_0$. *(Proposition 2.4.6)*

$\square$

Theorem 2.2.3. *(i)* $\mathsf{RCA}_0 \vdash (\mathit{S1} \to \mathit{S2})$

(ii) $\mathsf{RCA}_0 \vdash (\mathit{S2} \to \mathit{S3})$

(iii) $\mathsf{RCA}_0 \vdash (\mathit{S3} \to \mathit{S5})$

(iv) $\mathsf{RCA}_0 \vdash (\mathit{S1} \to \mathit{S4})$

(v) $\mathsf{RCA}_0 \vdash (\mathit{S4} \to \mathit{S5})$

Proof. (i) By definition.

(ii) By definition.

(iii) We prove the contrapositive. Suppose that T has a nonprincipal n-type $p = \{\psi_0(\bar{x}), \psi_1(\bar{x}), \ldots\}$. Then there are infinitely many $m \in M$ such that the formula

$$\theta_m = \bigwedge_{i<m} \psi_i \wedge \neg\psi_m$$

is consistent with T. These θ_m can be extended uniformly to an infinite coded sequence of distinct n-types.

(iv) Suppose that the property $(\mathsf{S1})$ holds of T, and we are given two models $\mathcal{A} \models T$ and $\mathcal{B} \models T$. We can construct an isomorphism $f : \mathcal{A} \to \mathcal{B}$ by an effective version of the usual back-and-forth argument. For an example of an effective back-and-forth argument, see the proof of Lemma 2.3.5 below.

(v) We prove the contrapositive. Suppose that T has a nonprincipal type p. By the Weak Completeness Theorem 2.1.3, there is a model $\mathcal{A}$ of T realizing p; and by the Type Omitting Theorem 2.1.9, there is a model $\mathcal{B}$ that does not realize p. These $\mathcal{A}$ and $\mathcal{B}$ cannot be isomorphic, so T is not $\aleph_0$-categorical.

□

Theorem 2.2.4. *All equivalences listed in the table are correct.*

Proof sketch. We have already proved many of these equivalences in Theorems 2.2.2 and 2.2.3. All others can be deduced from these. For example, we can see that $(\mathsf{S1} \to \mathsf{S5})$ holds in RCA_0 by combining parts (i), (ii), and (iii) of Theorem 2.2.3:

$$\mathsf{RCA}_0 \vdash (\mathsf{S1} \to \mathsf{S2}) \wedge (\mathsf{S2} \to \mathsf{S3}) \wedge (\mathsf{S3} \to \mathsf{S5})$$

and applying the rules of propositional logic. On the other hand, we can see that $(\mathsf{S5} \to \mathsf{S1})$ implies ACA_0 over RCA_0 by combining parts (i) and (ii) of Theorem 2.2.3 with part (iv) of Theorem 2.2.2:

$$\mathsf{RCA}_0 \vdash (\mathsf{S1} \to \mathsf{S2}) \wedge (\mathsf{S2} \to \mathsf{S3}) \wedge ((\mathsf{S5} \to \mathsf{S3}) \to \mathsf{ACA}_0).$$

The remaining directions are similar. □

We can also combine parts of Theorems 2.2.2 and 2.2.3 to show that the two remaining directions, $(\mathsf{S4} \to \mathsf{S1})$ and $(\mathsf{S4} \to \mathsf{S2})$, each imply WKL_0 over RCA_0. Hence their strength over RCA_0 lies somewhere between WKL_0 and ACA_0. The question of their precise strength remains open.

Question 2.2.5. What is the strength over RCA_0 of ($\mathsf{S4} \to \mathsf{S1}$) and ($\mathsf{S4} \to \mathsf{S2}$)?

There are other statements besides ($\mathsf{S1}$)–($\mathsf{S5}$) which are commonly given as pieces of the Ryll-Nardzewski theorem. Here we list a few statements that are provably equivalent, in RCA_0, to one of ($\mathsf{S1}$)–($\mathsf{S5}$). Some of these will be useful in the work that follows.

($\mathsf{S3}'$) For each n there is a number k such that any set $\{\phi_0, \ldots, \phi_k\}$ of n-ary formulas contains a pair ϕ_i, ϕ_j, $i \neq j$, such that $T \vdash \phi_i \leftrightarrow \phi_j$.

($\mathsf{S5}'$) Every model of T is *atomic*, i.e., realizes only principal types.

($\mathsf{S5}''$) There is an atomic model of T realizing all types of T.

Theorem 2.2.6. *(i)* RCA_0 *proves that a complete theory T has only finitely many n-types if and only if there is a number k such that any set $\{\phi_0, \ldots, \phi_k\}$ of n-ary formulas contains a pair ϕ_i, ϕ_j, $i \neq j$, such that $T \vdash \phi_i \leftrightarrow \phi_j$. In particular,* $\mathsf{RCA}_0 \vdash (\mathsf{S3} \leftrightarrow \mathsf{S3}')$.

(ii) $\mathsf{RCA}_0 \vdash (\mathsf{S5} \leftrightarrow \mathsf{S5}')$ *and* $\mathsf{RCA}_0 \vdash (\mathsf{S5} \leftrightarrow \mathsf{S5}'')$.

□

2.2.2 Reverse mathematics and theories with finitely many models

Recall our second question of basic model theory:

Q2. *For what finite values n may we have a complete theory with exactly n models up to isomorphism?*

In the classical setting, this question was settled by work of Ehrenfeucht and work of Vaught. Ehrenfeucht's idea was to add to a linear order a sequence of constant symbols that together give a small number of nonprincipal types, which can either be realized or omitted to give a certain number of nonisomorphic models. This can be carried out in ACA_0.

Theorem 2.2.7 (Classical and ACA_0. Ehrenfeucht)**.** *For every $n \geq 3$, there is a complete theory T with exactly n models up to isomorphism.*

Proof. See Chang and Keisler [7, Ex. 2.3.16]. □

Vaught's idea was, given a complete theory T which is not $\aleph_0$-categorical, to use the nonprincipal type guaranteed by the Ryll-Nardzewski Theorem 2.2.1 to show that T has at least three models. This can also be carried out in ACA_0:

Theorem 2.2.8 (Classical and ACA_0. Vaught)**.** *There is no complete theory with exactly two models up to isomorphism.*

Proof. See Chang and Keisler [7, Thm 2.3.15]. □

Since RCA_0 is enough to prove the Weak Completeness Theorem 2.1.3 and to prove that some theories are $\aleph_0$-categorical—for instance, the theory of dense linear orders without endpoints—we now have a full answer to Q2 over ACA_0:

Corollary 2.2.9 (Classical and ACA_0)**.** *Fix $n \geq 1$. There is a complete theory T with exactly n models up to isomorphism if and only if $n = 1$ or $n \geq 3$.*

It is not immediately clear whether Ehrenfeucht's and Vaught's constructions should work in systems weaker than ACA_0. In §2.7 below, we get a different answer to Q2 in the system $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$ by adapting a construction of Millar [47] from effective mathematics. Millar's idea was to define a complete decidable theory T with a recursive nonprincipal 1-type $p(x)$ such that there is exactly one decidable model omitting p and exactly $n-1$ decidable models realizing p, both up to classical and up to recursive isomorphism. This construction can be carried out assuming the failure of Weak König's Lemma:

Theorem 2.2.10 ($\mathsf{RCA_0} + \neg\mathsf{WKL_0}$). *For every $n \geq 1$, there is a complete theory with exactly n models up to isomorphism.*

Proof. See §2.7.3 below. $\square$

Corollary 2.2.11. *(i) $\neg\mathsf{WKL_0}$ implies the statement of Ehrenfeucht's Theorem 2.2.7 over $\mathsf{RCA_0}$.*

(ii) The statement of Vaught's Theorem 2.2.8 implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$.

It remains to answer $\mathsf{Q2}$ in the system $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. A reasonable first step is to ask whether the proofs of Corollary 2.2.9 or Theorem 2.2.10 can be carried out in this system. The work in §2.5 below gives the following:

Theorem 2.2.12. *Over $\mathsf{RCA_0}$, the following are equivalent:*

(i) $(\neg\mathsf{WKL_0}) \vee \mathsf{ACA_0}$

(ii) There is a complete theory with a nonprincipal type and only finitely many models up to isomorphism.

(iii) There is a complete theory with infinitely many n-types, for some n, and with only finitely many models up to isomorphism.

Proof. The direction $(i \to ii)$ follows from the use of a nonprincipal type in the proofs of Theorem 2.2.7 and Theorem 2.2.10 in the systems $\mathsf{ACA_0}$ and $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$, respectively. The direction $(ii \to iii)$ is immediate. The final direction $(iii \to i)$ follow from Proposition 2.5.7 below. $\square$

Although Theorem 2.2.12 is interesting in itself—it is the first example of a natural-seeming statement equivalent to $(\neg\mathsf{WKL_0}) \vee \mathsf{ACA_0}$ or, in its negation, to $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$—it is a serious obstacle if we want a full answer to $\mathsf{Q2}$ over $\mathsf{RCA_0}$. Since the constructions of Ehrenfeucht, Vaught, and Millar each require a nonprincipal type, Theorem 2.2.12 tells us none of them can be used in the system $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. Beyond this, we know very little about the case of $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$.

Question 2.2.13. Fix a model $(M,)$ of $\mathsf{WKL}_0+\neg\mathsf{ACA}_0$. Is there a complete theory $T\in$ with a finite number $n\in M$, $n\geq 2$ of models? If so, what values of n are possible?

2.3 Coding an extendable binary tree as a theory

Our first and most straightforward technique is one that has seen heavy use in effective mathematics, and has already been used in reverse mathematics by Hirschfeldt, Shore, and Slaman [30] and by Harris [25]. The earliest published use appears to be Ehrenfeucht [13].

Recall that we are working within a model $(M,)$ of RCA_0, and that $2^{<M}$ denotes the set of all finite binary strings. We say that a binary tree $\mathcal{T}\subseteq 2^{<M}$ is *extendable* if, for every $\sigma\in\mathcal{T}$, at least one of $\sigma^\frown 0,\sigma^\frown 1$ is in $\mathcal{T}$. (Here the $^\frown$ symbol denotes concatenation.) Fix an extendable binary tree $\mathcal{T}$, and let $L=(U_i)_{i\in M}$ be a relational language with each U_i unary. In §2.3.1 below we describe a complete L-theory T with the property that, for each $\sigma\in 2^{<M}$,

$$\begin{aligned}\sigma \text{ is in } \mathcal{T} \quad & \text{if and only if } T\vdash(\exists x)\left[\bigwedge_{\substack{i<|\sigma|\\ \sigma(i)=0}}\neg U_i(x)\wedge\bigwedge_{\substack{j<|\sigma|\\ \sigma(j)=1}}U_j(x)\right]\\ & \text{if and only if } T\vdash(\exists^{\geq n}x)\left[\bigwedge_{\substack{i<|\sigma|\\ \sigma(i)=0}}\neg U_i(x)\wedge\bigwedge_{\substack{j<|\sigma|\\ \sigma(j)=1}}U_j(x)\right] \text{ for all } n.\end{aligned}$$

The theory T also has quantifier elimination, so its 1-types are determined entirely by literals of the form $U_i(x)$ and $\neg U_i(x)$. This gives a natural correspondence between the 1-types of T and the paths in $\mathcal{T}$, and between the n-types of T and the coded tuples of paths in $\mathcal{T}$.

We give the full construction in §2.3.1, some basic verification in §2.3.2, and a direct application in §2.3.3. Further applications are obtained in §3.5, where we examine a specific instance of this construction.

2.3.1 Construction

Let $L = (U_i)_{i \in M}$ be a relational language with every U_i unary. Fix an extendable tree $\mathcal{T}$. (*Extendable* is defined at the beginning of this section.) Consider the following axiom schemes:

Ax I. $(\exists^{\geq n} x)\left[\bigwedge_{\substack{i<|\sigma| \\ \sigma(i)=0}} \neg U_i(x) \wedge \bigwedge_{\substack{j<|\sigma| \\ \sigma(j)=1}} U_j(x)\right]$ for every $n \in M$ and every $\sigma \in \mathcal{T}$.

Ax II. $\neg(\exists x)\left[\bigwedge_{\substack{i<|\sigma| \\ \sigma(i)=0}} \neg U_i(x) \wedge \bigwedge_{\substack{j<|\sigma| \\ \sigma(j)=1}} U_j(x)\right]$ for every $\sigma \notin \mathcal{T}$.

Let T^* be the collection of all sentences in Ax I and II, and let T be the deductive closure of T^*. This completes the construction. Although T^* is clearly in the second-order part of $(M,)$ by Δ^0_1 comprehension, it is not immediately evident that T is in . One of our first tasks in the next subsection is to prove that it is.

2.3.2 Verification

Here we list some important properties of T, such as its existence, completeness, and consistency. The analogous situation in effective mathematics is described in Harizanov [24, Section 7]. Unfortunately, we cannot rely on the proofs there, since in RCA_0 we do not have access to tools such as strong forms of the Completeness Theorem. Instead we give longer, elementary proofs.

Lemma 2.3.1 (RCA_0)**.** *T^* has effective quantifier elimination.*

Proof. Fix a quantifier-free L-formula $\phi(\bar{x}, y)$ which is a conjunction of literals. It suffices by Lemma 2.1.7 to show an effective procedure producing a quantifier-free ψ such that $T \vdash \psi \leftrightarrow (\exists y)\phi(\bar{x}, y)$. By identifying and renaming variables if necessary, we may assume that no conjunct in ϕ is of the form $y = x_i$ or $x_i = y$.

Check whether there is a $\sigma \in \mathcal{T}$ such that $|\sigma| \geq i$ and $\sigma(i) = 0$ whenever $\neg U_i(y)$ is a conjunct in ϕ, and $|\sigma| \geq i$ and $\sigma(i) = 1$ whenever $U_i(y)$ is in ϕ. If there is no such σ, then ϕ contradicts Ax II, so we may let ψ be the formal logical symbol Fa.

Now suppose there is such a σ, and let ψ be the formula obtained from ϕ by replacing each conjunct mentioning y with the propositional symbol Tr. Clearly $T^* \vdash (\exists y)\phi(\bar{x}, y) \to \psi(\bar{x})$. We wish to show the converse. Fix $n = |\bar{x}| + 1$. The following is a version of the Pigeonhole Principle, and is easily seen to be a tautology:

$$\left(\psi(\bar{x}) \wedge \bigwedge_{k<\ell<n} y_k \neq y_\ell\right) \to \left(\psi(\bar{x}) \wedge \bigvee_{k<n} \bigwedge_{i<n-1} y_k \neq x_i\right).$$

As ϕ has no conjunct of the form $y = x_i$ or $x_i = y$, we deduce a second tautology:

$$\left(\psi(\bar{x}) \wedge \bigwedge_{k<n} \left(\bigwedge_{\ell \neq k} y_k \neq y_\ell \wedge \bigwedge_{\substack{i<|\sigma| \\ \sigma(i)=0}} \neg U_i(y_k) \wedge \bigwedge_{\substack{j<|\sigma| \\ \sigma(j)=1}} U_j(y_k)\right)\right) \to \bigvee_{k<n} \phi(\bar{x}, y_k).$$

This statement, together with the instance of Ax I which uses the n and σ specified above, gives $T^* \vdash \psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$. □

Proposition 2.3.2 (RCA_0). *(i) For every L-sentence ϕ, either ϕ is provable from T^*, or $\neg\phi$ is provable from T^*.*

(ii) T is an element of .

(iii) T is a complete theory. T has quantifier elimination.

Proof. (i) Given an L-sentence ϕ, use the procedure from Lemma 2.3.1 to produce a quantifier-free ψ such that $T^* \vdash \phi \leftrightarrow \psi$. Since L is relational, ψ is a propositional combination of Tr and Fa, and hence provably equivalent either to Tr or to Fa. If Tr, then ϕ is in T; if Fa, then $\neg\phi$ is in T.

(ii) If T contains a contradiction, that is, a pair of sentences of the form ϕ and $\neg\phi$, then T is the set of all L-sentences, which is certainly in . Otherwise, by part (i), T contains exactly one of each pair $\{\phi, \neg\phi\}$: we can effectively decide which by searching for the shortest proof of either $T^* \vdash \phi$ or $T^* \vdash \neg\phi$.

(iii) Completeness of T follows from part (i). Quantifier elimination is inherited from T^*.

□

Lemma 2.3.3 (RCA_0)**.** *T is consistent.*

Proof. We build a model $\mathcal{A} \models T$ with domain $\{a_0, a_1, \ldots\}$, beginning with its quantifier-free diagram. For each $i, k \in M$, let $R_k(a_i)$ hold in $\mathcal{A}$ if and only if $\mathsf{left}(\sigma_i)(k) = 1$, where $\mathsf{left}(\sigma_i)$ is the path in $\mathcal{T}$ extending σ which is leftmost with respect to the ordering $0 < 1$. Recursively extend to a full quantifier-free diagram by adding formulas of the form $\neg\phi$ and $\phi \wedge \psi$, in the usual way. It is straightforward to check that this diagram satisfies every axiom in T^*. (Here we are using the usual truth-functional semantics, as given in Simpson [63, Ch. II.8].)

Now we extend to a complete diagram for $\mathcal{A}$. Fix any $\phi(\bar{a})$, where ϕ is a formula and $\bar{a}$ is a tuple of elements. We must decide whether to place $\phi(\bar{a})$ into the diagram of $\mathcal{A}$. By iterating the effective construction of Proposition 2.3.1, obtain a quantifier-free ψ such that $T^* \vdash \psi \leftrightarrow \phi$. Add $\phi(\bar{a})$ if and only if $\psi(\bar{a})$ is in the quantifier-free diagram. We claim that this process yields a complete, consistent diagram. For a contradiction, suppose that it does not. Then there is a formula $\phi(\bar{a})$ which fails to have one of the following properties:

- If $\phi(\bar{a}) = \neg\theta(\bar{a})$, then ϕ is in the diagram iff $\theta(\bar{a})$ is not in the diagram.
- If $\phi(\bar{a}) = \theta_0(\bar{a}) \wedge \theta_1(\bar{a})$, then $\phi(\bar{a})$ is in the diagram iff both $\theta_0(\bar{a})$ and $\theta_1(\bar{a})$ are in the diagram.
- If $\phi(\bar{a}) = (\forall x)\theta(\bar{a}, x)$, then $\phi(\bar{a})$ is in the diagram iff $\theta(\bar{a}, a_i)$ is in the diagram for every a_i.

But this is impossible by $\mathsf{I}\Sigma^0_1$ and the proof of Proposition 2.3.1. □

Lemma 2.3.4 (RCA_0). *(i) The 1-types of T correspond to paths in $\mathcal{T}$ in the following manner. If $p(x)$ is a 1-type of T, define a function $f_p : M \to \{0,1\}$ by $f_p(n) = 1 \iff U_n(x) \in p(x)$. The function f_p is a path in $\mathcal{T}$, and for every path f in $\mathcal{T}$, there is a unique 1-type $p(x)$ such that $f = f_p$.*

(ii) An n-type $p(x_0, \ldots, x_{n-1})$ is uniquely determined by the 1-types induced on its entries. In particular, the correspondence from (i) can be extended to a correspondence between n-types and coded n-tuples $\langle f_0, \ldots, f_{n-1}\rangle$ of paths in $\mathcal{T}$.

Proof. (i) By construction and the fact that T has quantifier elimination.

(ii) By construction, since the language L consists only of unary relations.

□

2.3.3 Applications

Recall from §2.2.1 the statements:

($\mathsf{S3}$) T has only finitely many n-types, for each n.

($\mathsf{S4}$) T is $\aleph_0$-categorical.

The construction given in §2.3.1 is enough to show one direction of Theorem 2.2.2(v):

Proposition 2.3.5. *Over RCA_0, the implication ($S4 \to S3$) implies WKL_0.*

Proof. We prove the contrapositive statement that, if WKL_0 fails, there is a theory T satisfying ($\mathsf{S4}$) but not ($\mathsf{S3}$). Let $\mathcal{T}_0$ be an infinite binary tree with no infinite path. Let $\langle \sigma_0, \sigma_1, \ldots\rangle$ be a one-to-one enumeration of all terminal nodes in $\mathcal{T}_0$. Define a second tree $\mathcal{T}$ by

$$\mathcal{T} = \mathcal{T}_0 \cup \{\sigma_i ^\frown 0^j : i, j \in M\}.$$

Then $\mathcal{T}$ is an extendable tree. (*Extendable* is defined at the beginning of §2.3.) Let T be the theory obtained from $\mathcal{T}$ using the construction of §2.3.1. By Lemma 2.3.4, each path in $\mathcal{T}$ corresponds to a unique 1-type of T. Since $\mathcal{T}$ has infinitely many paths, T has infinitely many distinct 1-types, and so does not satisfy (S3).

On the other hand, each 1-type p of T corresponds to a path f_p in $\mathcal{T}$ of the form $f_p = \sigma_i {}^\frown 0^M$ for some terminal node σ_i of $\mathcal{T}_0$. This σ_i, in turn, is associated with a formula

$$\bigwedge_{\substack{j<|\sigma_i| \\ \sigma_i(j)=0}} \neg U_j(x) \wedge \bigwedge_{\substack{j<|\sigma_i| \\ \sigma_i(j)=1}} U_j(x)$$

which generates p. Hence there is a procedure mapping every 1-type to a formula which generates it. With Lemma 2.3.4(iii), this gives a procedure for mapping any type of any arity to a formula generating it.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are two models of T, with domains $\{a_0, \ldots\}$ and $\{b_0, \ldots\}$, respectively. We now produce an isomorphism from $\mathcal{A}$ to $\mathcal{B}$:

Stage 0. Let f_0 be the empty function.

Odd stages $2s+1$. Suppose that f_{2s} is a finite partial elementary map from $\mathcal{A}$ into $\mathcal{B}$ with domain of size $2s$, enumerated $\langle a_{k_0}, \ldots, a_{k_{2s-1}} \rangle$. Let i be least such that a_i is not in the domain of f_{2s}. Use the procedure outlined above to find a formula $\phi(x_0, \ldots, x_{2s})$ generating $\mathsf{tp}^{\mathcal{A}}(a_{k_0}, \ldots, a_{k_{2s-1}}, a_i)$. Since f_{2s} is a partial elementary map, we know that

$$\mathsf{tp}^{\mathcal{A}}(a_{k_0}, \ldots, a_{k_{2s-1}}) = \mathsf{tp}^{\mathcal{B}}(f_{2s}(a_{k_0}), \ldots, f_{2s}(a_{k_{2s-1}})),$$

and in particular that there exists a b_j not in $\{f_{2s}(a_{k_0}), \ldots, f_{2s}(a_{k_{2s-1}})\}$ and such that $\mathcal{B} \models \phi(f_{2s}(a_{k_0}), \ldots, f_{2s}(a_{k_{2s-1}}), b_j)$. Let j be the least index of such a b_j, and define $f_{2s+1} = f_{2s} \cup \{(a_i, b_j)\}$.

Even stages $2s+2$. Let $\langle a_{k_0}, \ldots, a_{k_{2s}} \rangle$ be an enumeration of the domain of f_{2s+1}. Beginning with the least index j such that b_j is not in the range of f_{2s+1}, perform a procedure similar to the one given for odd stages to find the least index i such that a_i is not in the domain of f_{2s+1} and such that $\mathsf{tp}^{\mathcal{A}}(a_{k_0}, \ldots, a_{k_{2s}}, a_i) = \mathsf{tp}^{\mathcal{B}}(f_{2s+1}(a_{k_0}), \ldots, f_{2s+1}(a_{k_{2s}}), b_j)$. Let $f_{2s+2} = f_{2s+1} \cup \{(a_i, b_j)\}$.

Then Δ^0_1 comprehension allows us to form the limit $f = \bigcup_{s \in M} f_s$. It is straightforward to check that f is an isomorphism. □

The strategy we used to build f in the proof of Proposition 2.3.5 is called an *effective back-and-forth argument*.

2.4 A theory with infinitely many 1-types, whose every nonprincipal type computes K^Z

Recall that we work in a model $(M,)$ of RCA_0. Fix a set $Z \in$. We begin by constructing an infinite ternary tree $\mathcal{T} \subseteq \{0, 1, b\}^{<M}$ with infinitely many isolated paths and whose every nonisolated path computes the Turing jump K^Z. We then convert $\mathcal{T}$ into a theory T, and show that T has infinitely many 1-types and that K^Z is Δ^0_1 definable in each nonprincipal type of T. This allows us, in §2.4.3, to prove some directions of Theorem 2.2.2. Our construction is similar to some in the literature, for instance, Millar [48].

2.4.1 Construction

We define the set $\mathcal{T} \subseteq 2^{<M}$ as follows. Suppose that σ is any string in $\{0, 1, b\}^{<M}$ not beginning with b. Then σ can be written uniquely in the form

$$\sigma = i_0 {}^\frown b^{t_0} {}^\frown i_1 {}^\frown \cdots {}^\frown b^{t_{m-1}} {}^\frown i_m {}^\frown b^{t_*},$$

with $i_k \in \{0,1\}$, $t_k \in M$ for each k, and $t_* \in M$. We let σ be in $\mathcal{T}$ if and only if the following condition holds:

$$\text{For each } k < m,\ t_k \text{ is the least number } \geq k \text{ s.t. } i_0{}^\frown \cdots {}^\frown i_k = K^Z_{t_k} \restriction (k+1). \quad (2.1)$$

This completes the construction of $\mathcal{T}$. Before constructing the theory T, we point out that $\mathcal{T}$ is indeed a nonempty extendable tree:

Lemma 2.4.1 (RCA_0). *(i) The empty string $\emptyset$ is in $\mathcal{T}$.*

(ii) If $\sigma \subseteq \tau$ and $\tau \in \mathcal{T}$, then $\sigma \in \mathcal{T}$.

(iii) If $\sigma \in \mathcal{T}$, then $\sigma{}^\frown b \in \mathcal{T}$.

Proof. All three claims are immediate. $\square$

Now we code $\mathcal{T}$ as a binary tree $\mathcal{T}_0$ by defining a function $F : \{0,1,b\}^{<M} \to \{0,1\}^{<M}$:

$$\begin{aligned} F(\emptyset) &= \emptyset, \\ F(\sigma{}^\frown 0) &= F(\sigma){}^\frown 0{}^\frown 0, \\ F(\sigma{}^\frown 1) &= F(\sigma){}^\frown 0{}^\frown 1, \\ F(\sigma{}^\frown b) &= F(\sigma){}^\frown 1{}^\frown 0, \end{aligned}$$

and letting $\mathcal{T}_0 = \{\tau : \tau \subseteq F(\sigma) \text{ for some } \sigma \in \mathcal{T}\}$. Let T be the theory obtained from $\mathcal{T}_0$ by the method of §2.3.1. This completes the construction.

2.4.2 Verification

We claim that T has infinitely many 1-types, and we claim that K^Z is Δ^0_1 definable in every nonprincipal type of T. By Lemma 2.3.4, the 1-types of T correspond to paths in $\mathcal{T}_0$, which can be identified naturally with paths in $\mathcal{T}$. We may therefore rephrase the claim that T has infinitely many 1-types as part (ii) of the following lemma.

Lemma 2.4.2 (RCA_0). *(i) Given $\sigma \in \mathcal{T}$, we have $\sigma ^\frown 0 \in \mathcal{T} \iff \sigma ^\frown 1 \in \mathcal{T}$.*

(ii) The tree $\mathcal{T}$ has infinitely many paths.

Proof. (i) Immediate from the definition.

(ii) Let $\langle \sigma_0, \sigma_1, \ldots \rangle$ be a one-to-one enumeration of all strings in $\mathcal{T}$ that end in a 1. (There are infinitely many such σ_i.) We know by Lemma 2.4.1(iii) that $\mathcal{T}$ is extendable, so we may effectively extend every $\sigma \in \mathcal{T}$ to the leftmost path $\mathsf{left}(\sigma) \in \{0, 1, b\}^M$ of $\mathcal{T}$ extending σ, using the ordering $0 < 1 < b$. Then the coded sequence $\langle \mathsf{left}(\sigma_0), \mathsf{left}(\sigma_1), \ldots \rangle$ is a sequence of paths through $\mathcal{T}$. Since the mapping from σ_i to $\mathsf{left}(\sigma_i)$ is effective, this coded sequence exists in by Δ^0_1 comprehension. It is easy to see that $i \neq j$ implies $\mathsf{left}(\sigma_i) \neq \mathsf{left}(\sigma_j)$, so $\langle \mathsf{left}(\sigma_0), \ldots \rangle$ is a list of infinitely many distinct paths, as desired.

□

It remains to show that K^Z is Δ^0_1 definable in each nonprincipal type of T. This requires a few more facts about $\mathcal{T}$.

Lemma 2.4.3 (RCA_0). *(i) A path f through $\mathcal{T}$ is isolated if and only if f is of the form $f = \sigma ^\frown b^M$ for some finite string σ.*

(ii) K^Z is Δ^0_1 definable in each nonisolated path through $\mathcal{T}$.

(iii) If $\langle f_0, \ldots, f_{n-1} \rangle$ is a tuple of isolated paths through $\mathcal{T}$, then there is a level $\ell \in M$ above which every f_i is isolated.

Proof. (i) For the 'if' direction, suppose that $f = \sigma ^\frown b^M$, with $\sigma = i_0 ^\frown b^{t_0} ^\frown \cdots ^\frown b^{t_{m-2}} ^\frown i_{m-1}$. If there is no $t \geq k$ such that $K^Z_t \restriction m = i_0 ^\frown i_1 ^\frown \cdots ^\frown i_{m-1}$, then f is isolated above σ. If there is such a t, then f is isolated above $\sigma ^\frown b^{t+1}$.

For the 'only if' direction, we show the contrapositive. Suppose that f is a path through $\mathcal{T}$ such that $f(m) \in \{0, 1\}$ for infinitely many m. By

Lemma 2.4.2(ii), for each such m, the string $\sigma = (f \upharpoonright m)^\frown(1 - f(m))$ is in $\mathcal{T}$, and hence there is a path $g_m \neq f$ with $g_m \upharpoonright (m+1) = (f \upharpoonright m)^\frown(1-f(m))$. Since these m are cofinal in M, it follows that f is not isolated.

(ii) Suppose that f is an infinite path through $\mathcal{T}$ not ending in a string of b's. Such an f may be written

$$f = i_0 {}^\frown b^{t_0} {}^\frown i_1 {}^\frown b^{t_1} {}^\frown \cdots ,$$

with $i_k \in \{0, 1\}$ for every k. For every $s \in M$, the initial segment $\sigma_s \subseteq f$ given by

$$\sigma_s = i_0 {}^\frown b^{t_0} {}^\frown \cdots {}^\frown b^{t_{s-1}} {}^\frown i_s$$

is an element of $\mathcal{T}$. It follows from the definition of $\mathcal{T}$ that, for all $m \in M$:

$$(\forall s > t_{m-1}) \left[i_0 {}^\frown \cdots {}^\frown i_{m-1} = K^Z_s \upharpoonright m \right].$$

In other words, $i_0 {}^\frown \cdots {}^\frown i_{m-1} = K^Z \upharpoonright m$. This gives a Δ^0_1 definition for K^Z.

(iii) Let $\langle f_0, \ldots, f_{n-1} \rangle$ be a coded n-tuple of isolated paths in $\mathcal{T}$. By part (i), each f_j can be written in the form:

$$f_j = i_{j,0} {}^\frown b^{t_{j,0}} {}^\frown \cdots {}^\frown b^{t_{j,m_j-1}} {}^\frown i_{j,m_j-1} {}^\frown b^M.$$

The induction axioms of RCA_0 are not strong enough, at least on their face, to guarantee the existence of the tuple $\langle m_j : j < n \rangle$. This adds to the complexity of our proof.

Every f_j, being isolated, falls into one or more of the following cases:

1. f_j has an initial segment of the form $i_{j,0} {}^\frown b^{t_{j,0}} {}^\frown \cdots {}^\frown i_{j,m} {}^\frown b^{s+1}$ with $s \geq m$ and such that $i_{j,0} {}^\frown \cdots {}^\frown i_{j,m} = K^Z_s \upharpoonright (m + 1)$.

2. There is a k such that $i_{j,k} = 0$ while $K^Z(k) = 1$.

3. There is a k such that $i_{j,k} = 1$ while $K^Z(k) = 0$.

Whether f_j falls into case 1 is a Σ^0_1 question, and case 2, also a Σ^0_1 question. Use bounded Σ^0_1 comprehension to partition the indices $j < n$ along these lines:

$$X_1 = \{j < n : f_j \text{ falls into case } 1\},$$

$$X_2 = \{j < n : j \notin X_1 \text{ and } f_j \text{ falls into case } 2\},$$

$$X_3 = \{j < n : j \notin X_1 \cup X_2\}.$$

Then every element of X_3 falls into case 3. It suffices to show that for each $z \in \{1, 2, 3\}$ there is a level ℓ_z above which f_j is isolated for all $j \in X_z$, and take $\ell = \max(\ell_1, \ell_2, \ell_3)$. First consider $z = 1$. Assign to each $j \in X_1$ a string $\sigma_j \subseteq f_j$ as in the statement of case 1. Then f_j is isolated above the length $|\sigma_j|$. Let ℓ_1 be the maximum of $|\sigma_j|$ as j ranges over X_1.

Now consider $z = 2$. For all $j \in X_2$, the formula $(\exists k \exists s)[i_{j,k} = 0 \text{ and } K^Z_s(k) = 1]$ holds. Use Σ^0_1 bounding to assign to each $j \in X_2$ a pair k_j, s_j witnessing this. Choose any $\sigma_j \subseteq f_j$ of the form

$$\sigma_j = i_{j,0} {}^\frown b^{t_{j,0}} {}^\frown \cdots {}^\frown i_{j,k_j} {}^\frown \tau {}^\frown b^{s_j+1}$$

where τ is a string. Then f_j is isolated above the length $|\sigma_j|$. Let ℓ_2 be the maximum of $|\sigma_j|$ as j ranges over X_2.

Lastly, consider $z = 3$. Since it is a Π^0_1 question to ask whether two paths are equal, we may assume by bounded Π^0_1 comprehension that the paths f_j are all distinct as j ranges over X_3. Let $j_0, j_1 \in X_3$ be distinct elements, and consider the paths f_{j_0}, f_{j_1}. Let k be least such that $i_{j_0,k} \neq i_{j_1,k}$; we may assume by symmetry that $i_{j_0,k} = 0$ and $i_{j_1,k} = 1$. Then $K^Z(k)$ must equal 0, since otherwise j_0 would be an element of X_2. Let $\sigma_{j_1} = i_{j_1,0} {}^\frown b^{t_{j_1,0}} {}^\frown \cdots {}^\frown i_{j_1,k}$. It follows that f_{j_1} is isolated above $|\sigma_{j_1}|$. Repeat this procedure on pairs from $X_3 - \{j_1\}$, and so on, until there is a σ_j associated to all but one element of X_3, say j'. Let $\sigma_{j'}$ be such that $f_{j'}$ is isolated above $|\sigma_{j'}|$, and let ℓ_3 be the maximum of $|\sigma_j|$ as j ranges over X_3.

Now $\ell = \max(\ell_1, \ell_2, \ell_3)$ is the desired bound.

□

This is enough to verify the last desired property:

Proposition 2.4.4 (RCA_0). *K^Z is Δ^0_1 definable in each nonprincipal type of T.*

Proof. Let $p(x_0, \ldots, x_{n-1})$ be a nonprincipal n-type for some n. Since the language of T consists only of unary relations, p may be decomposed into 1-types $\langle p_0, \ldots, p_{n-1}\rangle$:

$$p(x_0, \ldots, x_{n-1}) \iff p_0(x_0), \ldots, p_{n-1}(x_{n-1}).$$

The 1-types $\langle p_0, \ldots, p_{n-1}\rangle$ correspond to a tuple $\langle f_0, \ldots, f_{n-1}\rangle$ of paths through $\mathcal{T}$. Since p is nonprincipal, there is an i such that f_i is nonisolated by Lemma 2.4.3(iii). Therefore K^Z is Δ^0_1 definable from f_i, and hence from p, by Lemma 2.4.3(ii). □

2.4.3 Applications

Recall from §2.2.1 the statements:

($\mathsf{S3}$) T has only finitely many n-types, for each n.

($\mathsf{S4}$) T is $\aleph_0$-categorical.

($\mathsf{S5}$) All types of T are principal.

We use this section's construction to prove two parts of Theorem 2.2.2, beginning with part (iv):

Proposition 2.4.5. *Over* RCA_0*, the implication* ($\mathsf{S5} \to \mathsf{S3}$) *implies* ACA_0*.*

Proof. Suppose that ($\mathsf{S5} \to \mathsf{S3}$) holds, and fix any set $Z \in$. Let T be the theory constructed in §2.4.1. Since T has infinitely many 1-types, T satisfies ($\neg\mathsf{S3}$). Then T satisfies ($\neg\mathsf{S5}$), i.e., T has a nonprincipal type p. By Proposition 2.4.4 above, K^Z is Δ^0_1 definable from p, and so K^Z exists by Δ^0_1 comprehension. Since Z was arbitrary, we conclude by Lemma 2.1.2 that ACA_0 holds. □

Next, we prove Theorem 2.2.2(vi):

Proposition 2.4.6. *Over* RCA_0*, the implication* ($\mathsf{S5} \to \mathsf{S4}$) *implies* ACA_0*.*

Proof. Fix any set $Z \in$, and let T be the theory constructed in §2.4.1. It is enough to exhibit two models $\mathcal{A}, \mathcal{B}$ of T such that K^Z is Δ^0_1 definable in any isomorphism $f : \mathcal{A} \to \mathcal{B}$. Let $\langle \sigma_0, \sigma_1, \ldots \rangle$ be a one-to-one enumeration of all strings in the tree $\mathcal{T}_0$. For each σ_i, let $\mathsf{left}(\sigma_i)$ be the leftmost path of $\mathcal{T}_0$ extending σ_i; similarly, let $\mathsf{right}(\sigma_i)$ be the rightmost path extending σ_i. We may form the coded sequences $\langle \mathsf{left}(\sigma_0), \mathsf{left}(\sigma_1), \ldots \rangle$ and $\langle \mathsf{right}(\sigma_0), \mathsf{right}(\sigma_1), \ldots \rangle$ by Δ^0_1 comprehension.

First we build the model $\mathcal{A}$, with domain $\{a_0, a_1, \ldots\}$. For each $i, k \in M$, let $R_k(a_i)$ hold in $\mathcal{A}$ if and only if $\mathsf{left}(\sigma_i)(k) = 1$. It is easy to check that $\mathcal{A}$ satisfies the axioms of §2.3.1 semantically. Fill in the rest of the diagram as in the proof of Lemma 2.3.3 so that $\mathcal{A}$ is a model of T. Build a second model $\mathcal{B}$ with domain $\{b_0, b_1, \ldots\}$ by a similar method: for each $i, k \in M$, let $R_k(b_i)$ hold in $\mathcal{B}$ if and only if $\mathsf{right}(\sigma_i)(k) = 1$, and fill in the rest of the diagram.

Now, suppose that $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism. Use f to define a function $g : M \to M$ by $g(i) = j$ whenever $f(a_i) = b_j$. Then $\mathsf{left}(\sigma_i) = \mathsf{right}(\sigma_{g(i)})$ for all $i \in M$. In particular either $\sigma_i \subseteq \sigma_{g(i)}$ or $\sigma_i \supseteq \sigma_{g(i)}$, and the longer of the two, which we denote by $\sigma_i \cup \sigma_{g(i)}$, is isolated in $\mathcal{T}_0$. It follows that σ_i is isolated if and only if there is no string τ such that $\sigma_i \subseteq \tau \subseteq \sigma_i \cup \sigma_{g(i)}$, and such that both $\tau^\frown 0$ and

$\tau^\frown 1$ are elements of $\mathcal{T}_0$. This gives a uniform procedure for deciding whether a given σ is isolated, and, in particular, allows us to define a nonisolated path of $\mathcal{T}_0$, and hence a nonisolated path of $\mathcal{T}$. By Lemma 2.4.3(ii) and Δ^0_1 comprehension, the Turing jump K^Z is an element of . We conclude by Lemma 2.1.2 that ACA_0 holds. $\square$

2.5 Models from a tree of Henkin constructions

For the following informal discussion, we reason in WKL_0. Fix a set $Z \in$, a language L, a complete L-theory T with infinitely many n-types for some n, and a model $\mathcal{A} \models T$ with domain $A = \{a_0, a_1, \ldots\}$. We produce a second model $\mathcal{B} \models T$ with domain $B = \{b_0, b_1, \ldots\}$ such that the Turing jump K^Z is Δ^0_1 definable in any elementary embedding $f : \mathcal{B} \to \mathcal{A}$. We achieve this by making the function $g : M \to M$ defined by $g(m) = n \iff f(b_m) = a_n$ grow roughly as fast as the *modulus function* of K^Z, which is given by $m \mapsto \min\{s > m : K^Z_s \restriction m = K^Z \restriction m\}$. More specifically, we ensure that, if m is an element of $K^Z_{at\ s}$, there is an n-ary formula satisfied in $\mathcal{B}$ by an n-tuple taken from the initial segment $\{b_0, b_1, \ldots, b_{2n(m+1)-1}\}$ of B, but not in $\mathcal{A}$ by any n-tuple from the initial segment $\{a_0, \ldots, a_{s-1}\}$ of A. Then if $f : \mathcal{B} \to \mathcal{A}$ is an elementary embedding, the function given by $m \mapsto \max\limits_{i<2n(m+1)} g(i)$ bounds the modulus function of K^Z.

The model $\mathcal{B}$ itself is obtained by the following method. We construct a binary tree $\mathcal{H}^*$ such that any node $\sigma \in \mathcal{H}^*$ of length s represents the first s-many steps of a Henkin-style construction, and such that the construction along any infinite path of $\mathcal{H}^*$ yields a model $\mathcal{B}$ with the property outlined above. We then show that $\mathcal{H}^*$ is infinite, and apply Weak König's Lemma to obtain $\mathcal{B}$.

2.5.1 Construction

We begin with some definitions. Fix a language L and a complete, consistent L-theory T.

Definition 2.5.1. (i) Let L' be the enriched language $L \cup \{c_0, c_1, \ldots\}$, where each c_i is a constant symbol not in L. Let $\langle \phi_s \rangle_s$ be a one-to-one enumeration of all L'-sentences. First, define a $2^{<M}$-indexed sequence $\langle D_\sigma \rangle_{\sigma \in 2^{<M}}$ of sets of L'-sentences by

$$D_\sigma = \{\phi_s : s < |\sigma| \text{ and } \sigma(s) = 1\} \cup \{\neg\phi_s : s < |\sigma| \text{ and } \sigma(s) = 0\}.$$

Second, define a sequence $\langle W_s \rangle_{s \in M}$ of sets of L'-sentences by recursion:

$$\begin{aligned} W_0 &= \emptyset \\ W_{s+1} &= \begin{cases} W_s \cup \{\phi_s \to \psi(c_{2k+1})\} \text{ if } \phi_s \text{ is of the form } (\exists\, x)\psi(x), \\ \qquad \text{where } 2k+1 \text{ is the least odd index such that } c_{2k+1} \\ \qquad \text{is not mentioned in } W_s \text{ or in any } D_\sigma \text{ with } |\sigma| \le s. \\ W_s \ \text{ if } \phi_s \text{ is not of this form.} \end{cases} \end{aligned}$$

Third, define a tree $\mathcal{H} \subseteq 2^{<M}$ by

$$\mathcal{H} = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \text{ is consistent}\}.$$

We call $\mathcal{H}$ the *full tree of odd Henkin diagrams*. (‘Odd’ because we are using only the odd-numbered constants to witness existential sentences.)

(ii) Given an infinite path β in $\mathcal{H}$, let $D_\beta = \bigcup_{s \in M} D_{\beta \restriction s}$. Then D_β is a complete, consistent L'-theory. Define an equivalence relation E on the constants $\{c_0, c_1, \ldots\}$ by $c_i E c_j$ iff $D_\beta \vdash c_i = c_j$. Denote the E-equivalence class of c_i by $[c_i]_E$, and let $\langle b_0, b_1, \ldots \rangle$ be the one-to-one listing of all E-equivalence classes given by

$$b_m = [c_{i_m}]_E, \text{ where } i_m \text{ is least s.t. } c_{i_m} \notin b_k \text{ for all } k < m.$$

Let $\mathcal{B}$ be the L-structure such that, for any L-formula ϕ,

$$\mathcal{B} \models \phi(b_0, \ldots, b_{n-1}) \iff D_\beta \vdash \phi(c_{i_0}, \ldots, c_{i_{m-1}}).$$

Then $\mathcal{B}$ is a model of T. We say that $\mathcal{B}$ is the *Henkin model encoded by* β.

Now fix a model $\mathcal{A}$ of T. We define an infinite subtree $\mathcal{H}^* \subseteq \mathcal{H}$ of the full tree of odd Henkin diagrams such that, if β is an infinite path of $\mathcal{H}^*$ and $\mathcal{B}$ is the Henkin model encoded by β, then K^Z is Δ^0_1 definable in any elementary embedding $f : \mathcal{B} \to \mathcal{A}$. Then WKL_0 ensures that such a path β exists, giving the desired model $\mathcal{B}$.

For each $t \in M$, choose an n-ary L-formula $\theta_t(\bar{x})$ such that $T \vdash (\exists \bar{x})\theta_t(\bar{x})$, and such that θ_t is not satisfied by any tuple taken from $\{a_0, \ldots, a_t\}$ in $\mathcal{A}$. (This is possible by Theorem 2.2.6(i), since T has infinitely many n-types.) For each $s \in M$, define a finite set D^*_s of L'-sentences:

$$D^*_s = \{\theta_t(c_{2mn}, c_{2mn+2}, \ldots, c_{2(m+1)n-2}) : m, t < s \text{ and } m \in K^Z_{at\ t}\}.$$

Note that $D^*_s \subseteq D^*_{s+1}$ for each s. Define the subtree $\mathcal{H}^*$ of $\mathcal{H}$ by:

$$\mathcal{H}^* = \left\{\sigma \in 2^{<M} : T \cup D_\sigma \cup D^*_{|\sigma|} \cup W_{|\sigma|} \text{ is consistent}\right\}. \tag{2.2}$$

This completes the construction.

2.5.2 Verification

There are two facts to verify: first, that $\mathcal{H}^*$ is infinite, and second, if a model $\mathcal{B}$ is encoded by a path in $\mathcal{H}^*$, then K^Z is Δ^0_1 definable in any elementary embedding of $\mathcal{B}$ into $\mathcal{A}$.

Lemma 2.5.2 (RCA_0)**.** *The tree $\mathcal{H}^*$ is infinite.*

Proof. Fix any $s \in M$. It suffices to show that $\mathcal{H}^*$ has an element of length s. We may choose a finite tuple $\langle c_i^{\mathcal{A}} : i < N\rangle$ of elements of $\mathcal{A}$ such that $(\mathcal{A}, c_i^{\mathcal{A}} : i < N)$ is a model of $T \cup D_s^* \cup W_s$. In particular, $\langle c_i^{\mathcal{A}} : i < N\rangle$ contains all constants mentioned in $\phi_0, \ldots, \phi_{s-1}$, where $\langle \phi_t \rangle_t$ is the enumeration of all L'-sentences fixed in Definition 2.5.1(i). Define a string σ of length s by

$$\sigma(t) = \begin{cases} 1 \text{ if } (\mathcal{A}, c_i^{\mathcal{A}} : i < N) \models \phi_t, \\ 0 \text{ otherwise} \end{cases}$$

for all $t < s$. Then $(\mathcal{A}, c_i^{\mathcal{A}} : i < N)$ is a model of $T \cup D_\sigma \cup D_s^* \cup W_s$, so $T \cup D_\sigma \cup D_s^* \cup W_s$ is consistent. Therefore σ is in $\mathcal{H}^*$, as desired. $\square$

Lemma 2.5.3 (RCA_0)**.** *If $\mathcal{B}$ is the model encoded by an infinite path β in $\mathcal{H}^*$, and $f : \mathcal{B} \to \mathcal{A}$ is an elementary embedding, then K^Z is Δ^0_1 definable from f.*

Proof. Suppose that $\mathcal{B}$ is the model encoded by some path β in $\mathcal{H}^*$, and that $f : \mathcal{B} \to \mathcal{A}$ is an elementary embedding. Define a mapping $h : M \to M$ by

$$h(m) = \text{greatest } j \text{ s.t. } f([c_{2mn+2i}]_E) = a_j \text{ for some } i < n.$$

By the definition of D_s^*, if there is a t such that $m \in K_t^Z$, then $m \in K_{h(m)}^Z$. Hence we have $m \in K^Z \iff m \in K_{h(m)}^Z$, which gives a Δ^0_1 definition for K^Z. $\square$

2.5.3 Applications

Recall from §2.2.1 the statements:

($\mathsf{S3}$) T has only finitely many n-types for each n.

($\mathsf{S4}$) T is $\aleph_0$-categorical.

We say that a model $\mathcal{A}$ of a theory T is *elementary-universal* if, for any model $\mathcal{B}$ of T, there is an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$. The construction in §2.5.1 above is tailored to give the following result.

Lemma 2.5.4. $\mathsf{WKL_0} + \neg\mathsf{ACA_0} \vdash$ *('T has an elementary-universal model'*$\rightarrow$ *S3*).

Proof. Suppose that $(M,)$ is a model of $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. By Lemma 2.1.2, we may fix a set $Z \in$ whose Turing jump K^Z is not in . We show that the contrapositive statement ($\neg$S3 $\rightarrow$'T has no elementary-universal model') holds in $(M,)$.

Fix a complete theory $T \in$ with infinitely many n-types, and fix a model $\mathcal{A} \in$ of T. Use the construction of §2.4.1 and Lemma 2.5.3 to obtain a second model $\mathcal{B} \in$ of T such that K^Z is Δ^0_1 definable in every elementary embedding from $\mathcal{B}$ into $\mathcal{A}$. This means, by our choice of Z, that no $f \in$ can be an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$. In particular, $\mathcal{A}$ is not elementary-universal. □

Since any model of an $\aleph_0$-categorical theory is elementary-universal, the following is an immediate consequence of Lemma 2.5.4.

Lemma 2.5.5. $\mathsf{WKL_0} + \neg\mathsf{ACA_0} \vdash$ (*S4* $\rightarrow$ *S3*).

We are ready to prove the remaining direction of Theorem 2.2.2(v), the other having been proved in Proposition 2.3.5 above.

Proposition 2.5.6. $\mathsf{WKL_0} \vdash$ (*S4* $\rightarrow$ *S3*)

Proof. We know from Lemma 2.5.5 that $\mathsf{WKL_0} + \neg\mathsf{ACA_0} \vdash$ (S4 $\rightarrow$ S3). On the other hand, as noted in §2.2.1, $\mathsf{ACA_0}$ is sufficiently strong to carry out the usual proof of equivalence of all the principles (S1) through (S5), and in particular $\mathsf{ACA_0} \vdash$ (S4 $\rightarrow$ S3). Hence we conclude that $\mathsf{WKL_0} \vdash$ (S4 $\rightarrow$ S3). □

The construction from this section also justifies an assertion in §2.2.2. The following proposition completes the proof of Theorem 2.2.12:

Proposition 2.5.7. *Over* $\mathsf{WKL_0}$*, the following are equivalent:*

(i) $\mathsf{ACA_0}$

(ii) There is a complete theory with a nonprincipal type and only finitely many models.

(iii) There is a complete theory with infinitely many n-types for some n, and only finitely many models.

Proof. Reason in $\mathsf{WKL_0}$. The implication ($i \to ii$) follows from the use of a nonprincipal type in the proof of Ehrenfeucht's Theorem 2.2.7 in the system $\mathsf{ACA_0}$. The implication ($ii \to iii$) is immediate from the definitions.

We prove the final implication ($iii \to i$) by way of its contrapositive statement ($\neg i \to \neg iii$). Suppose that $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$ holds, and let T be a complete theory with infinitely many n-types for some n. Dovetail the proof of Lemma 2.5.4 to get a coded sequence $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ of models of T such that no $\mathcal{A}_j$ embeds elementarily into any $\mathcal{A}_i$ with $i < j$. (For each triple $\langle i, j, m \rangle$ where $i < j$, if m is in $K^Z_{at\ s}$, use the method of §2.5.1 to ensure that there is a formula realized by a tuple from among the first $2n(\langle i, j, m \rangle + 1)$-many elements of $\mathcal{A}_j$ but not by any tuple from among the first s-many elements of $\mathcal{A}_i$.) We have produced an infinite list of pairwise nonisomorphic models of T, so (iii) fails, as desired. □

2.6 Theories with only finitely many n-types for every n

The *Ryll-Nardzewski function* for a theory T is the Σ^0_2 partial function RN_T from M to M given by:

$$\begin{aligned} \mathrm{RN}_T(n) = m \quad &\iff \quad T \text{ has exactly } m \text{ different } n\text{-types} \\ &\iff \quad \text{there exists a sequence } \phi_0, \ldots, \phi_{m-1} \text{ of } n\text{-ary formulas} \\ &\qquad\quad \text{such that } T \vdash \phi_0 \vee \cdots \vee \phi_{m-1} \text{ and } T \nvdash \phi_i \to \phi_j \text{ for each} \\ &\qquad\quad i \neq j, \text{ and for all } n\text{-ary } \psi \text{ and all } i \text{ s.t. } T \vdash \psi \to \phi_i \text{ we} \\ &\qquad\quad \text{have } T \vdash \phi_i \to \psi. \end{aligned}$$

If $\mathrm{RN}_T(n)$ has no value according to the above definition, we treat $\mathrm{RN}_T(n)$ as an infinite number. The properties ($\mathsf{S1}$), ($\mathsf{S2}$), and ($\mathsf{S3}$) from §2.2.1 can all be phrased in terms of RN_T.

In this section, we prove several directions of Theorem 2.2.2 by constructing examples of a theory T for which RN_T is finite-valued, but for which RCA_0 cannot prove the existence of RN_T. One of these examples, given in Proposition 2.6.12, has a RN_T so fast-growing that ACA_0 is needed to prove even that RN_T is dominated by a function in the second-order part of $(M,)$. A second example, given in the proof of Proposition 2.6.11 and used again in that of Proposition 2.6.13, has a RN_T that is slow-growing, but whose existence nonetheless implies ACA_0. Our theories are built using a simple common framework, given in §2.6.1 below, which takes as a parameter a coded sequence $\langle X_1, X_2, \ldots\rangle$ of sets. By varying this parameter, we control RN_T.

In effective model theory, similar constructions have been done before to control the Turing degree of RN_T for a decidable $\aleph_0$-categorical theory with infinitely many predicates (Palyutin [53] and Venning [71, Ch. 2]) and with a single binary predicate (Herrmann [26], Schmerl [59], and Venning [71, Ch. 3]). Both our construction and our verification are very similar to Palyutin's, when done carefully in second-order arithmetic. Our construction is also similar to Venning's [Ch. 2], but the verification more elementary.

2.6.1 Construction

Let L be the language $L = \langle R^n_s\rangle_{s\in M, n\geq 1}$, with each R^n_s an n-ary relation. Let $\langle X_1, X_2, \ldots\rangle$ be a coded sequence of sets. We introduce three axiom schemes:

Ax I. $R^n_s(x_0, \ldots, x_{n-1}) \to x_i \neq x_j$, for each n, s and each pair $i, j < n$ with $i \neq j$.

Ax II. $\neg R^n_s(\bar{x})$, for each n, s such that $s \notin X_n$.

Ax III. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L'-literals, where $L' = \{R^n_s : n, s < \ell\}$ for some $\ell > |\bar{x}| + 1$;
- For every atomic L'-formula θ with variables in $\bar{x}$, either θ or $\neg\theta$ appears as a conjunct in ψ;
- $\phi(\bar{x}, y)$ is consistent with Ax I and II;
- Every conjunct in ψ is a conjunct in ϕ;

Let T^* denote the collection of all sentences in Ax I–III, and let T be the deductive closure of T^*. This completes the construction. Notice that we have not yet proved the existence either of T^* or of T in the second-order part of $(M,)$. For T^*, this follows from Lemma 2.6.2 below, where we prove that the consistency check in Ax III can be performed effectively. For T, existence is proved in Proposition 2.6.5 using quantifier elimination.

The intuition behind these axioms is as follows. Axiom I is an n-ary version of the irreflexivity property for binary relations: R^n_s holds only of n-tuples whose entries are all distinct. This limits the number of quantifier-free formulas that may hold of an n-tuple. Axiom II relates the parameter $\langle X_1, X_2, \ldots\rangle$ to the number of different quantifier-free formulas that might hold of an n-tuple. Axiom III then binds this number to $\mathrm{RN}_T(n)$ by providing quantifier elimination.

2.6.2 Verification

Most of this section is devoted to checking that the T defined in §2.6.1 is an element of , is complete, and is consistent. The exception is Lemma 2.6.10, in which we relate the coded sequence $\langle X_1, X_2, \ldots\rangle$ to the Ryll-Nardzewski function RN_T. The following technical lemma will be useful in this section, and again in §2.7.

Lemma 2.6.1 (RCA_0). *Let $L_0 = \langle Q_n\rangle_n$ be a relational language. Let $\Psi = \{\psi_s : s \in M\}$ be L_0-theory where each ψ_s is of the form $(\forall \bar{x}, \bar{y})[\ell_s(\bar{x}) \vee \theta_s(\bar{x}, \bar{y})]$ where θ_s is quantifier-free and ℓ_s is either $Q_n(\bar{x})$ or $\neg Q_n(\bar{x})$, where $n \geq s$ and Q_n is not mentioned in any ψ_t, $t < s$. Then there is a procedure that decides, given a quantifier-free L-formula $\phi(\bar{z})$, whether $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent.*

Proof. Fix a quantifier-free formula $\phi(z_0, \ldots, z_{m-1})$. Let n be the greatest index such that Q_n is mentioned in ϕ, and consider the set $\Psi_n = \{\psi_s : s \leq n\}$. Recall that a theory is *consistent* if does not entail a contradiction. We claim that $\Psi \cup \{(\exists \bar{z})\phi(\bar{x})\}$ is consistent if and only if $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{x})\}$ has an m-element model. We prove this claim by a series of implications:

(a) If $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent, then $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent.

(b) If $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent, then $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model.

(c) If $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model, then $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model.

(d) If $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model, then $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent.

Item (a) is immediate. For item (b), notice that it is possible to construct a propositional formula P such that if $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent then P is consistent, and if P is satisfiable then $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model. (Use one propositional variable to represent the truth value of each relevant ψ_s on each tuple taken from $\bar{z}$.) Item (c) holds because, given an m-element model of $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$, we can effectively transform it into a model of $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ by reassigning the truth values of ℓ_s to satisfy ψ_s for each $s > n$. Item (d) follows from the Soundness Theorem, which is provable in $\mathsf{RCA_0}$—see Simpson [63, Theorem II.8.8].

Our procedure works as follows: Given a formula $\phi(z_0, \ldots, z_{m-1})$, find n as above, and construct the propositional formula P used in (b). Test all truth valuations to see whether P is consistent. If so, $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent. If not, $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is inconsistent. $\square$

Lemma 2.6.2 ($\mathsf{RCA_0}$). *There is a procedure to check whether a quantifier-free L-formula ϕ is consistent with Axioms I and II.*

Proof. We may rewrite Axiom I by replacing the $\rightarrow$ with an equivalent $\vee$, and restricting the parameters n, s so as not to conflict with Axiom II:

$$\neg R^n_s(x_0, \ldots, x_{n-1}) \vee x_i \neq x_j, \text{ for each } n, s \in M \text{ such that } s \in X_n \text{ and each pair } i, j < n \text{ such that } i \neq j.$$

Then, after an appropriate reindexing of the relations R^n_s, our axioms meet the hypothesis of Lemma 2.6.1. The result follows. $\square$

Recall that T^* denotes the collection of all sentences in Ax I–III. We are ready to begin dealing with T^* directly.

Lemma 2.6.3 ($\mathsf{RCA_0}$). *T^* is an element of .*

Proof. We can easily tell whether a given formula is in Ax I or Ax II. Lemma 2.6.2 gives a method for deciding whether or not a formula is in Ax III. $\square$

Lemma 2.6.4 ($\mathsf{RCA_0}$). *The theory T^* has effective quantifier elimination.*

Proof. By Lemma 2.1.7, it is enough to give an effective procedure that takes as input any conjunction of literals $\phi(\bar{x}, y)$ and returns a quantifier-free formula $\psi(\bar{x})$ such that $T^* \vdash (\exists y)\phi(\bar{x}, y) \leftrightarrow \psi(\bar{x})$. By performing the appropriate substitutions, we may assume that no literal in ϕ is of the form $(z_0 = z_1)$. First use the effective procedure given by Lemma 2.6.2 to see whether ϕ is consistent with Axioms I and II. If it is not, we conclude that $T^* \vdash (\exists y)\phi(\bar{x}, y) \leftrightarrow \mathsf{Fa}$.

If it is consistent, let $\psi(\bar{x})$ be the formula produced from ϕ by substituting Tr for each conjunct mentioning the variable y. Let $L' = \{R^n_s : n, s < \ell\}$, where ℓ is a number greater than any n or s such that R^n_s is mentioned in ψ. Use Lemma 2.6.2 to find all conjunctions $\psi_0, \psi_1, \ldots, \psi_m$ of L'-literals without repetitions such that

- $\psi_i \wedge \phi$ is consistent with Ax I and II.
- Every conjunct of ψ is a conjunct of ψ_i.
- For every atomic L'-formula θ with variables in $\bar{x}$, either θ or $\neg\theta$ appears as a conjunct in ψ_i.

Then $T^* \vdash (\exists y)\phi \rightarrow (\psi_0 \vee \cdots \vee \psi_m)$. The converse direction $T^* \vdash (\psi_0 \vee \cdots \vee \psi_m) \rightarrow (\exists y)\phi$ follows from Ax III applied to each pair $\phi, \phi \wedge \psi_i$. $\square$

Recall that T denotes the deductive closure of T^*.

Proposition 2.6.5 (RCA_0). *(i) For every L-sentence ϕ, either ϕ is provable from T^*, or $\neg\phi$ is provable from T^*.*

(ii) T is an element of .

(iii) T has quantifier elimination. T is a complete theory.

Proof. Similar to the proof of Proposition 2.3.2. $\square$

Next, we verify that T is consistent. It suffices to show that T has a model. This is achieved in Proposition 2.6.9 below, using an effective version of the Fraïssé limit construction. This argument is both clean and reusable—we use it again in the proof of Proposition 2.6.13 and later in §2.7—but requires some definitions and lemmas. The following definitions are based on those given by Csima, Harizanov, Miller, and Montalbán [10] for Fraïssé limits in recursive mathematics.

Definition 2.6.6. Fix a language L_0 of relation symbols. Let $\mathbb{K} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ be a sequence of finite L_0-structures.

(i) We say that $\mathbb{K}$ has the *effective hereditary property (EHP)* if there is a function that, given an index i and a finite set F of elements from $\mathcal{A}_i$, returns an index j and an isomorphism from $\mathcal{A}_j$ to the induced substructure $\mathcal{A}_i \upharpoonright F$.

(ii) We say that $\mathbb{K}$ has the *effective joint embedding property (EJEP)* if there is a function that, given indices $\langle i, j \rangle$, returns an index k and a pair of embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_k$ and $\mathcal{A}_j \hookrightarrow \mathcal{A}_k$.

(iii) We say that $\mathbb{K}$ has the *effective amalgamation property (EAP)* if there is a functions that, given indices $\langle i, j, k \rangle$ and injections $f : \mathcal{A}_i \to \mathcal{A}_j$ and $g : \mathcal{A}_i \to \mathcal{A}_k$, returns an index ℓ, an embedding $e : \mathcal{A}_j \hookrightarrow \mathcal{A}_\ell$, and an injection $h : \mathcal{A}_k \to \mathcal{A}_\ell$ such that $h \circ f = e \circ g$ and, if f and g are embeddings, h is an embedding as well.

(iv) Let $\mathcal{A}$ be a countably infinite L_0-structure with domain A. Suppose that there is a pair of functions h_0, h_1 such that h_0 maps finite subsets $F \subseteq A$ surjectively onto the indices $\{0, 1, \ldots\}$ of $\mathbb{K}$, and h_1 maps finite subsets $F \subseteq A$ to isomorphisms from the induced substructure $\mathcal{A} \upharpoonright F$ to $\mathcal{A}_{h_0(F)}$. Suppose further that, for every choice of a finite $F \subseteq A$, a pair of indices $\langle i, j \rangle$, an isomorphism f from $\mathcal{A} \upharpoonright F$ to $\mathcal{A}_i$, and an embedding $g : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$, there is a second finite $G \subseteq A$ containing F and an isomorphism from $\mathcal{A} \upharpoonright G$ to $\mathcal{A}_j$ which agrees with $g \circ f$ on F. Then we say that $\mathcal{A}$ is an *effective Fraïssé limit of* $\mathbb{K}$.

When interpreted in the standard model REC of RCA_0, the definitions of EHP, EJEP, and EAP agree with those of the *computable* hereditary, joint embedding, and amalgamation properties in [10]. Our notion of *effective Fraïssé limit* is essentially the same, except that we require an explicit mapping from finite substructures of $\mathcal{A}$ onto $\mathbb{K}$. (The same effect is achieved in [10] using what they call a *canonical age.*)

Lemma 2.6.7 (RCA_0)**.** *Let L_0 be a relational language, and let $\mathbb{K} = (\mathcal{A}_i)_{i\in M}$ be a sequence of finite L_0-structures. If $\mathbb{K}$ has the EHP, the EJEP, and the EAP, then $\mathbb{K}$ has an effective Fraïssé limit.*

Proof. Similar to [10, Thm 3.9]. □

Lemma 2.6.8 (RCA_0)**.** *Let L_0 be a relational language, and let T_0 be an L_0-theory axiomatized by a set T_0' of $\forall\exists$-sentences. Let $\mathbb{K} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots\rangle$ be a sequence of finite models of the $\forall$ part of T_0 with the EHP, the EJEP, and the EAP. Suppose that, for any $\exists$ L_0-formula $\phi(\bar{x})$ such that $(\forall \bar{x})\phi(\bar{x})$ is in T_0', and any $(\mathcal{A}_i, \bar{b})$ with $\bar{b}$ having the same length as $\bar{x}$, there is an $\mathcal{A}_j$ and an embedding $g : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ such that $\mathcal{A}_j \models \phi(g(\bar{b}))$. Then any effective Fraïssé limit of $\mathbb{K}$ is a model of T_0.*

Proof. Suppose that $\mathcal{A}$ is an effective Fraïssé limit of $\mathbb{K}$ with domain A. It suffices to show that $\mathcal{A}$ satisfies T_0'. Let ϕ be an n-ary $\exists$ formula such that $(\forall \bar{x})\phi(\bar{x})$ is in T_0'. Fix any n-tuple $\bar{a}$ taken from A, and let $F \subseteq A$ be a finite set containing all entries of $\bar{a}$. Using the functions h_0, h_1 from the definition of effective Fraïssé limit, find an index i and an isomorphism f from the induced substructure $\mathcal{A} \upharpoonright F$ to $\mathcal{A}_i$. By assumption, there is an $\mathcal{A}_j$ and an embedding $g : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ such that $\mathcal{A}_j \models \phi(g(f(\bar{a})))$. Use the definition of effective Fraïssé limit to get a finite $G \subseteq A$ containing F such that $\mathcal{A} \upharpoonright G$ embeds into $\mathcal{A}_j$ by a mapping agreeing with $g \circ f$ on F. Then $\mathcal{A} \upharpoonright G \models \phi(\bar{a})$, and hence $\mathcal{A} \models \phi(\bar{a})$. Since ϕ and $\bar{a}$ were arbitrary, $\mathcal{A}$ satisfies T_0', as desired. □

We are now ready to verify the consistency of the theory T.

Proposition 2.6.9 (RCA_0)**.** *T is consistent.*

Proof. Notice that the axioms for T given in §2.6.1 consist of $\forall\exists$ sentences. To see that T has a model, it is enough to construct a sequence $\mathbb{K} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ meeting the hypotheses of Lemmas 2.6.7 and 2.6.8 with T in place of T_0. We begin by defining $\mathbb{K}$, and then verify that $\mathbb{K}$ has the required properties.

Let Y be the set of all triples $\langle n, s, \sigma \rangle$, where n is a natural number and σ is a function mapping each tuple taken from $\{0, \ldots, n-1\}^{\leq n}$ to a value in $\{0,1\}^{s+1}$, with the property that, if $\bar{y}$ has a repeated entry, we have $\sigma(\bar{y})(t) = 0$ for all $t \leq s$. This Y is an element of by Δ^0_1 comprehension. Let G be a surjection $G : M \to Y$. Each $\mathcal{A}_i$ is constructed as follows. Suppose that $G(i) = \langle n, s, \sigma \rangle$. Let $\mathcal{A}_i$ be the L-structure with domain $\{a_0, \ldots, a_{n-1}\}$ such that, for all $1 \leq k \leq n$, all $t \leq s$, and all k-tuples $\langle j_0, \ldots, j_{k-1} \rangle$ taken from $\{0, \ldots, n-1\}$, we have

$$\mathcal{A}_i \models R^k_t(a_{j_0}, \ldots, a_{j_{k-1}}) \iff (t \in X_k \text{ and } \sigma(\langle j_0, \ldots, j_{k-1} \rangle)(t) = 1)\,,$$

and $\mathcal{A}_i \models \neg R^k_t(\bar{a})$ for all other $t, k, \bar{a}$.

It is clear from the definition that $\mathbb{K}$ has the EHP, the EJEP, and the EAP, and hence by Lemma 2.6.7 has an effective Fraïssé limit $\mathcal{A}$. It can be checked that $\mathbb{K}$ satisfies the hypothesis of Lemma 2.6.8, and hence $\mathcal{A}$ is a model of T. $\square$

We now show how the coded sequence $\langle X_1, X_2, \ldots \rangle$ relates to $\mathrm{RN}_T(n)$.

Lemma 2.6.10 (RCA_0)**.** *Define a function* $\mathbf{d}$ *on tuples* $\bar{a} = \langle a_1, \ldots, a_n \rangle \in M^{<M}$ *by:*

$$\mathbf{d}(\emptyset) = 1,$$

$$\mathbf{d}(\bar{a}) = \sum_{m=1}^{n} S(n,m) \prod_{k=1}^{m} 2^{\left(\frac{m!}{(m-k)!}\right) a_k}, \text{ whenever } |\bar{a}| \geq 1,$$

where $S(n,m)$ *is the number of ways to partition an* n*-element set into* m *nonempty subsets.*[2] *The following statements hold.*

[2]These $S(n,m)$ are called *Stirling numbers of the second kind.*

(i) If $\bar{a} = \langle a_1, \ldots, a_n \rangle$ and $\bar{b} = \langle b_1, \ldots, b_n \rangle$ are n-tuples such that $\mathbf{d}(a_1, \ldots, a_k) = \mathbf{d}(b_1, \ldots, b_k)$ for all $k \leq n$, then $\bar{a} = \bar{b}$.

(ii) If the tuple $\langle |X_1|, \ldots, |X_n| \rangle$ exists in M, then $\mathrm{RN}_T(n) = \mathbf{d}(|X_1|, \ldots, |X_n|)$.

(iii) If $\mathrm{RN}_T(n)$ is finite, then $\langle |X_1|, \ldots, |X_n| \rangle$ exists in .

(iv) The Ryll-Nardzewski function RN_T exists in if and only if the function $n \mapsto |X_n|$ exists in .

Proof. (i) This is immediate when $n = 0$. If $\langle a_0, \ldots, a_k \rangle = \langle b_0, \ldots, b_k \rangle$ and $\mathbf{d}(a_0, \ldots, a_{k+1}) = \mathbf{d}(b_0, \ldots, b_{k+1})$, then it is clear from the definition of $\mathbf{d}$ that $a_{k+1} = b_{k+1}$. The result now follows by Δ^0_1 induction.

(ii) If $n = 0$, then there is exactly one 0-type, namely T itself, so $\mathrm{RN}_T(0) = 1 = \mathbf{d}(\emptyset)$. The case when $n \geq 1$ follows by a straightforward induction.

(iii) It is clear that, for all $k \leq n$, we have $|X_k| \leq \mathrm{RN}_T(n)$. Using bounded Σ^0_1 comprehension we may form the set $\{\langle k, i \rangle : |X_k| \geq i \text{ and } k \leq n\}$, from which $\langle |X_1|, \ldots, |X_n| \rangle$ is Δ^0_1 definable.

(iv) The 'if' direction is immediate from part (ii). For the 'only if' direction, suppose RN_T is in , and fix n. We know by parts (i), (ii), and (iii) that $\langle |X_1|, \ldots, |X_n| \rangle$ is in , and is the unique n-tuple satisfying that $\mathrm{RN}_T(k) = \mathbf{d}(|X_1|, \ldots, |X_k|)$ for every $k \leq n$. Thus we can find $|X_n|$ by testing each n-tuple for this property.

□

2.6.3 Applications

Recall from §2.2.1 the statements:

(S1) There is a function f such that, for all n, T has exactly $f(n)$ distinct n-types.

(S2) There is a function f such that, for all n, T has no more than $f(n)$ distinct n-types.

(S3) T has only finitely many n-types, for each n.

We now use the construction of §2.6.1 to prove Theorem 2.2.2(i):

Proposition 2.6.11. *Over* RCA_0, *the implication* ($\mathsf{S2} \to \mathsf{S1}$) *implies* ACA_0.

Proof. Suppose that ($\mathsf{S2} \to \mathsf{S1}$) holds. Let Z be any set, and recall from Definition 2.1.1 the *Turing jump* K^Z and its enumeration $\langle K^Z_0, K^Z_1, \ldots \rangle$. Define sets $X_1, X_2, \ldots$ by, for each s, n,

$$s \in X_{n+1} \iff n \in K^Z_{at\ s}.$$

The coded sequence $\langle X_1, \ldots \rangle$ exists by Δ^0_1 comprehension. Let T be the theory constructed by the method of §2.6.1 using $\langle X_1, \ldots \rangle$ as its parameter. Since each X_n has size ≤ 1, we can see by Lemma 2.6.10(ii) that RN_T is dominated by the function $f(n) = \mathbf{d}(\underbrace{1, 1, \ldots, 1}_{n \text{ times}})$. Hence T satisfies ($\mathsf{S2}$). Since ($\mathsf{S2} \to \mathsf{S1}$) holds, T satisfies ($\mathsf{S1}$) as well, that is, RN_T is an element of . By Lemma 2.6.10(iv), the function $n \mapsto |X_{n+1}|$ is in as well. But this is the characteristic function of K^Z. We conclude by Lemma 2.1.2 that ACA_0 holds. $\square$

Next, we verify Theorem 2.2.2(iii):

Proposition 2.6.12. *Over* RCA_0, *the implication* ($\mathsf{S3} \to \mathsf{S2}$) *implies* ACA_0.

Proof. Suppose that ($\mathsf{S3} \to \mathsf{S2}$) holds. Fix any set Z. Define sets $X_1, X_2, \ldots$ by, for each $s, n \in M$,

$$s \in X_{n+1} \iff (\exists t)[t \leq s < 2t \wedge n \in K^Z_{at\ t}].$$

If $n \in K^Z_{at\ t}$ for some t, then $|X_{n+1}| = t$; if there is no such t, then $|X_{n+1}| = 0$. The coded sequence $\langle X_1, \ldots \rangle$ exists by Δ^0_1 comprehension. Let T be the theory constructed by the method of §2.6.1 using $\langle X_1, \ldots \rangle$ as its parameter.

For each $n \geq 1$, $K \upharpoonright n$ exists by bounded Σ^0_1 comprehension, so way may form the tuple $\langle |X_1|, \ldots, |X_n| \rangle$. It follows by Lemma 2.6.10(ii) that $\mathrm{RN}_T(n)$ is a finite number, and $K^Z \upharpoonright n = K^Z_{\mathrm{RN}_T(n)} \upharpoonright n$. Thus T satisfies (S3). Since (S3 $\to$ S2) holds, T satisfies (S2) as well. Let f be a function such that $f(n) \geq \mathrm{RN}_T(n)$ for all n. Then we have $K^Z \upharpoonright n = K^Z_{f(n)} \upharpoonright n$ for all n, so K^Z is in by Δ^0_1 comprehension. We conclude by Lemma 2.1.2 that ACA_0 holds. $\square$

Finally, we prove Theorem 2.2.2(ii). In fact, we prove a stronger result.

Proposition 2.6.13. *Over* RCA_0*, the implication (S2 $\to$ 'T has a prime model') implies* ACA_0*.*

Proof. Fix any set Z. Define a coded sequence of sets $\langle X_1, \ldots \rangle$ and a theory T as in the proof of Proposition 2.6.11 above. As we have seen, T satisfies (S2). We construct two models $\mathcal{A}, \mathcal{B}$ of T such that, if $\mathcal{C}$ is a third model, and $e_0 : \mathcal{C} \hookrightarrow \mathcal{A}$, $e_1 : \mathcal{C} \hookrightarrow \mathcal{B}$ are embeddings, then K^Z is computable from e_0 and e_1. The models $\mathcal{A}, \mathcal{B}$ will be the effective Fraïssé limits of sequences $\mathbb{K}_0$ and $\mathbb{K}_1$, respectively.

Let Y be the set of all pairs $\langle n, \sigma \rangle$ such that n is a natural number, and $\sigma : \{0, \ldots, n-1\}^{\leq n} \to \{0, 1\}$ is a function such that $\sigma(\bar{x}) = 0$ whenever $\bar{x}$ has a repeated entry. This Y is a recursive set. Let $G : M \to Y$ be an infinite-to-one surjection. We use G to define sequences $\mathbb{K}_0 = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ and $\mathbb{K}_1 = \langle \mathcal{B}_0, \mathcal{B}_1, \ldots \rangle$ of finite structures. If $G(i) = \langle n, \sigma \rangle$, then $\mathcal{A}_i$ has domain $\{a_0, \ldots, a_{n-1}\}$ and, for all s and all tuples $\langle j_0, \ldots, j_{k-1} \rangle \in \{0, \ldots, n-1\}^{\leq n}$ of length $k \geq 1$,

$$\mathcal{A}_i \models R^k_s(a_{j_0}, \ldots, a_{j_{k-1}}) \iff (s \in X_k \text{ and } \sigma(j_0, \ldots, j_{k-1}) = 1 \text{ and } i > s)\,,$$

and, for all other $s, k, \bar{a}$, we have $\mathcal{A}_i \models \neg R^k_s(\bar{a})$. The structure $\mathcal{B}_i$ has domain $\{b_0, \ldots, b_{n-1}\}$ and, for all s and all tuples $\langle j_0, \ldots, j_{k-1} \rangle \in \{0, \ldots, n-1\}^{\leq n}$,

$$\mathcal{B}_i \models R^k_s(b_{j_0}, \ldots, b_{j_{k-1}}) \iff (s \in X_k \text{ and } (\sigma(j_0, \ldots, j_{k-1}) = 1 \text{ or } i \leq s))\,,$$

and, for all other $s, k, \bar{b}$, we have $\mathcal{B}_i \models \neg R^k_s(\bar{b})$. The coded sequences $\mathbb{K}_0, \mathbb{K}_1$ exist by Δ^0_1 comprehension. It can be checked that $\mathbb{K}_0$ and $\mathbb{K}_1$ each have the EHP, the EJEP, and the EAP, and satisfy the hypotheses of Proposition 2.6.8. Hence, by Propositions 2.6.7 and 2.6.8, $\mathbb{K}_0$ has an effective Fraïssé limit $\mathcal{A} \models T$ and $\mathbb{K}_1$ has an effective Fraïssé limit $\mathcal{B} \models T$.

Now suppose that $\mathcal{C}$ is a model of T with domain C, and $e_0 : \mathcal{C} \hookrightarrow \mathcal{A}$, $e_1 : \mathcal{C} \hookrightarrow \mathcal{B}$ are embeddings. Given a finite $F \subseteq C$, we may use e_0 and the fact that $\mathcal{A}$ is an effective Fraïssé limit to find an index i and an isomorphism from the induced substructure $\mathcal{C} \upharpoonright F$ to $\mathcal{A}_i$. Likewise, we may use e_1 to find an index j and an isomorphism from $\mathcal{C} \upharpoonright F$ to $\mathcal{B}_j$, giving an isomorphism from $\mathcal{A}_i$ to $\mathcal{B}_j$.

Fix enumerations $\bar{a}$ of the elements of $\mathcal{A}_i$ and $\bar{b}$ of the elements of $\mathcal{B}_j$ such that $(\mathcal{A}_i, \bar{a}) \cong (\mathcal{B}_j, \bar{b})$. Let n be the cardinality of F, and suppose that $n \in K^Z$. Then there is an s such that $n \in K^Z_s$ and $X_{n+1} = \{s\}$. We claim that $s \leq \max(i, j)$. To see this, assume that $j < s$, so that $\mathcal{B}_j \models R^{n+1}_s(\bar{b})$ by construction of $\mathcal{B}_j$. Then $\mathcal{A}_i \models R^{n+1}_s(\bar{a})$ as well, which implies by construction of $\mathcal{A}_i$ that $i \geq s$. Our claim now proven, we deduce that n is in K^Z if and only if n is in $K^Z_{\max(i,j)}$. Hence K^Z exists by Δ^0_1 comprehension. We conclude by Lemma 2.1.2 that ACA_0 holds. □

Corollary 2.6.14. *Over* RCA_0*, the implication* ($\mathit{S2} \to \mathit{S4}$) *implies* ACA_0*.*

2.7 Theories with finitely many models

In this section, we present a construction due to Millar [47]. Given any $n \geq 2$, it builds a complete, decidable theory T with exactly n decidable models, both up to classical isomorphism and up to recursive isomorphism. We use this construction largely unchanged in the system $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$ to prove Theorem 2.2.10. The construction itself is given in §2.7.1 below. We begin with some definitions and an overview of our goals.

Definition 2.7.1. A *disjoint* Σ^0_1 *pair* is a coded sequence $\langle U_s, V_s\rangle_{s\in M}$ of pairs $U_s, V_s \subseteq M$ with the following properties:

- Each U_s and V_s is finite, with $\max(U_s \cup V_s) < s$.
- $U_s \cap V_s = \emptyset$ for every s.
- $U_s \subseteq U_{s+1}$ and $V_s \subseteq V_{s+1}$ for every s.

Given a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$, a set $C \subseteq M$ is called a *separating set for* $\langle U_s, V_s\rangle_s$ if, for every s, we have $U_s \subseteq C \subseteq (M - V_s)$. If no such C exists, then $\langle U_s, V_s\rangle_s$ is called an *inseparable* Σ^0_1 *pair*. The Σ^0_1 *separation principle* is the statement: There is no inseparable Σ^0_1 pair.

In the standard model REC of RCA_0, a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$ can be written as a pair of recursive approximations $\langle U_s\rangle_s$, $\langle V_s\rangle_s$ to disjoint r.e. sets $U = \lim_s U_s$ and $V = \lim_s V_s$. If $\langle U_s, V_s\rangle_s$ is an inseparable Σ^0_1 pair in REC, then the limits U and V are *recursively inseparable* in the sense of recursion theory.

We are interested in these pairs, first, because they figure in Millar's construction, and second, because of the following result of Friedman, Simpson, and Smith [19] pinpointing the reverse-mathematical complexity of the Σ^0_1 separation principle.

Lemma 2.7.2. $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow (\Sigma^0_1 \text{ separation})$

Proof. See Simpson [63, Lemma IV.4.4]. $\square$

Fix a natural number $n \geq 2$ and a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$. Our construction in §2.7.1 is of a complete, decidable theory T with the following properties:

1. T has exactly one nonprincipal 1-type $p(x)$.
2. For every $k < n$, T has a decidable model $\mathcal{A}$ with exactly k distinct elements realizing p.

3. For every $k \in M$, if $\mathcal{A}, \mathcal{B}$ are models of T each with exactly k distinct elements realizing p, then there is an isomorphism $f : \mathcal{A} \cong \mathcal{B}$ which is Δ^0_1 definable in $\mathcal{A} \oplus \mathcal{B}$.

4. If $\mathcal{A}$ is a model of T with at least n distinct elements realizing p, then there is a separating set C for $\langle U_s, V_s \rangle_s$ which is Δ^0_1 definable in $\mathcal{A}$.

If we are working within a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$ and $\langle U_s, V_s \rangle_s$ is an inseparable Σ^0_1 pair as given by Lemma 2.7.2, then the properties above imply that T has exactly n nonisomorphic models. (This is proved in §2.7.3 below.)

2.7.1 Construction

Fix a natural number $n \geq 2$ and a disjoint Σ^0_1 pair $\langle U_s, V_s \rangle_s$. Let $L = \langle P_s, R_s \rangle_{s \in M}$ be a language with every P_s unary and every R_s n-ary. Consider the following axiom schemes:

Ax I. $P_s(x) \to P_t(x)$, whenever $t \leq s$.

Ax II. $R_k(x_0, \ldots, x_{n-1}) \to \bigwedge_{i<j<n} (P_k(x_i) \land x_i \neq x_j)$

Ax III. $\left(\bigwedge_{i<j<n} (P_s(x_i) \land x_i \neq x_j)\right) \to R_k(x_0, \ldots, x_{n-1})$, whenever $k \in U_s$.

Ax IV. $\left(\bigwedge_{i<j<n} (P_s(x_i) \land x_i \neq x_j)\right) \to \neg R_k(x_0, \ldots, x_{n-1})$, whenever $k \in V_s$.

Ax V. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L'-literals, where $L' = \{P_s, R_s : s < \ell\}$ for some ℓ;
- For every atomic L'-formula θ with variables in $\bar{x}, y$, either θ or $\neg\theta$ appears as a conjunct in ψ;
- $\phi(\bar{x}, y)$ is consistent with Ax I–IV;

- Every conjunct in ψ is a conjunct in ϕ;

Let T^* be the collection of all sentences in Ax I–V, and let T be the deductive closure of T^*. This completes the construction. Notice that we have not yet established that either T^* or T is in . The existence of T^* is a consequence of Lemma 2.7.3 below, while that of T is part of Proposition 2.7.5.

The intuition behind these axioms is as follows. Given an element a of a model and an index s, the statement $P_s(a)$ is read as, 'a is turned on at stage s'. Axiom I says that the stages at which an element is turned on form an initial segment of M—possibly $\emptyset$ or all of M. Axiom II says that R_k can hold of a tuple $\bar{a}$ only if the entries of $\bar{a}$ are all distinct and are all turned on at stage k. Axioms III and IV together say that if $\bar{a}$ is a tuple of distinct elements, all turned on at stage s, then $U_s \subseteq \{k : R_k(\bar{a}) \text{ holds}\} \subseteq M - V_s$. As with the similar axiom in §2.6.1 above, Axiom V gives the theory effective quantifier elimination.

2.7.2 Verification

Lemma 2.7.3 (RCA_0)**.** *There is a procedure to decide whether a given L-formula ϕ is consistent with Axioms I–IV.*

Proof. Assume that $k \in U_s \cup V_s$ implies $k < s$. Combine Axioms I, III, and IV into a single equivalent scheme of the form:

$$P_s(x_0) \rightarrow \left(\bigwedge_{t \leq s} P_t(x_0) \wedge \left(\bigwedge_{i<j<n} P_s(x_i) \wedge x_i \neq x_j \right) \rightarrow \left(\bigwedge_{k \in U_s} R_k(x_0, \dots, x_{n-1}) \wedge \right. \right.$$
$$\left. \left. \bigwedge_{k \in V_s} \neg R_k(x_0, \dots, x_{n-1}) \right) \right).$$

As in the proof of Lemma 2.6.2, we may replace the initial $\rightarrow$ with $\vee$ in both this scheme and Axiom II and perform an appropriate reindexing of the relations to get a sequence of sentences satisfying the hypothesis of Lemma 2.6.1 above. The result follows. □

It follows that T^* is in .

Lemma 2.7.4 (RCA_0)**.** *The theory T^* has quantifer elimination.*

Proof. Similar to the proof of Lemma 2.6.4. □

Proposition 2.7.5. *T is in , is complete, and has quantifier elimination.*

Proof. Similar to the proof of Proposition 2.3.2. □

Lemma 2.7.6 (RCA_0)**.** *The theory T is consistent.*

Proof. It suffices by the Soundness Theorem to show that T has a model. Suppose that $\mathcal{A}$ is a finite L-structure, and suppose that there is an s_0 such that, for every $s \geq s_0$, every n-tuple $\bar{a}$ of elements of $\mathcal{A}$, and every entry a_i of $\bar{a}$, we have $\mathcal{A} \models \neg P_s(a_i)$ and $\mathcal{A} \models \neg R_s(\bar{a})$. Then there is a recursive procedure to check whether $\mathcal{A}$ is a model of Axioms I–IV. Let $\mathbb{K}$ be an infinite-to-one enumeration of all finite L-structures which have such an s_0 and which are consistent with Axioms I–IV. This $\mathbb{K}$ satisfies the hypotheses of Lemma 2.6.8, and hence, by Lemmas 2.6.7 and 2.6.8, has an effective Fraïssé limit which is a model of T. □

Lemma 2.7.7 (RCA_0)**.** *T has exactly one nonprincipal 1-type $p(x)$. Furthermore, $P_s(x)$ is in $p(x)$ for every s, and if $q(x)$ is a 1-type of T not equal to $p(x)$, then there is an s such that $\neg P_s(x) \in q$.*

Proof. As in Harizanov [24, Lemma 10.7]. □

Lemma 2.7.8 (RCA_0)**.** *For every $k < n$, T has a decidable model $\mathcal{A}$ with exactly k distinct elements realizing p.*

Proof. Use a Fraïssé construction similar to that in the proof of Lemma 2.7.6, except, instead of just one, allow up to k distinct elements to realize p. □

Lemma 2.7.9 (RCA_0)**.** *Fix a number $k < n$ and models $\mathcal{A}, \mathcal{B}$ of T. If $\mathcal{A}$ and $\mathcal{B}$ each have exactly k distinct elements realizing p, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. An effective back-and-forth argument. □

Lemma 2.7.10 (RCA_0)**.** *If $\mathcal{A}$ is a model of T with at least n distinct elements realizing p, then there is a separating set C for $\langle U_s, V_s \rangle_s$. In particular, $\langle U_s, V_s \rangle_s$ is not an inseparable Σ^0_1 pair.*

Proof. Suppose $\mathcal{A}$ is such a model, and let $\bar{a}$ be a tuple of distinct elements all realizing p. Define $C = \{k : \mathcal{A} \models R_k(\bar{a})\}$. Then Ax III ensures that $U_s \subseteq C$ for all s, and Ax IV ensures $V_s \subseteq M - C$ for all s. Therefore, C is a separating set for $\langle U_s, V_s \rangle_s$. □

2.7.3 Application

We now prove the remaining theorem from §2.2.2.

Proof of Theorem 2.2.10. Assume WKL_0 fails. When $n = 1$, use the $\aleph_0$-categorical theory constructed in the proof of Proposition 2.3.5. (Alternatively, we could use an effectively $\aleph_0$-categorical theory such as the theory of dense linear orders without endpoints.) Now suppose $n \geq 2$. Lemma 2.7.2 tells us that there is an inseparable Σ^0_1 pair $\langle U_s, V_s \rangle_s$. Let T be the theory constructed by the method of §2.7.1 using $\langle U_s, V_s \rangle_s$ and the given n. Lemmas 2.7.8, 2.7.9, and 2.7.10 together imply that T has exactly n models up to isomorphism. □

CHAPTER 3

FURTHER RESULTS IN THE REVERSE MATHEMATICS OF MODEL THEORY

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3.1 Introduction

We consider the reverse mathematics of basic model theory. The corresponding study in effective mathematics, called interchangeably *effective*, *recursive*, or *computable model theory*, is well developed at this point, and the subject of surveys [24, 34] and monographs [2]. While Simpson and others have long since formalized the basics of first-order logic in second-order arithmetic, only recently have researchers such as Harris, Hirschfeldt, Lange, and Shore begun the wholesale formalization of model-theoretic theorems. Most of these theorems turn out to be equivalent to one of RCA_0, WKL_0, or ACA_0—three of the familiar Big Five systems—or to an induction principle such as $\mathsf{I}\Sigma^0_2$. Some theorems fall into other, previously unknown complexity classes. For example, Hirschfeldt, Shore, and Slaman [30] isolated new classes by considering the existence theorem for atomic models and type omitting theorems; the in the previous Chapter, we presented a model-theoretic statement equivalent over RCA_0 to $\mathsf{ACA}_0 \vee \neg\mathsf{WKL}_0$; and in the present paper, we introduce a family of statements equivalent to $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$. Still other theorems reveal new classes not directly through their statements but through a careful study of their proofs. This was the case for the hierarchies of genericity principles $\Pi^0_n\mathsf{G}$ and $\Pi^0_n\mathsf{GA}$ found by Hirschfeldt, Lange, and Shore [28].

In this paper, we focus on existence theorems for countable homogeneous models (related to work in [28]), existence theorems for countable saturated models, theorems concerning elementary embeddings (building on work in the previous

Chapter), theorems concerning type amalgamation properties (again related to [28]), and some other well-known theorems such as the existence of order indiscernibles. We separate our results into five categories along these lines and summarize them separately in §3.2.1, §3.2.2, §3.2.3, §3.2.4, and §3.2.5, respectively.

Most of the theorems we analyze have the expected complexities of $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, $\mathsf{ACA_0}$, or, echoing the previous Chapter, $\neg\mathsf{WKL_0} \vee \mathsf{ACA_0}$. Breaking the pattern are several more unusual theorems; the most striking is a statement equivalent to the disjunction $\mathsf{WKL_0} \vee \mathsf{I}\Sigma^0_2$ over $\mathsf{RCA_0}$ (see Theorems 3.2.24 and 3.2.14.) We know of only one other natural statement with this complexity: Friedman, Simpson, and Yu [20] have shown that $\mathsf{WKL_0} \vee \mathsf{I}\Sigma^0_2$ holds if and only if any iteration f^n of a continuous function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is itself continuous. In our case, the theorem is provable by an induction argument (using $\mathsf{I}\Sigma^0_2$) or by a compactness argument (using $\mathsf{WKL_0}$). But neither of these is the optimal proof from a reverse-mathematical standpoint—rather, the optimal proof appears simply to choose between them based on what axioms are available.

3.1.1 Conventions and organization

All definitions are in the language of second-order arithmetic. Unless otherwise specified, all infinite sets are countable, all reasoning is carried out in $\mathsf{RCA_0}$, and all theorems are theorems of $\mathsf{RCA_0}$. We use the symbols $(M,)$ to represent a model of $\mathsf{RCA_0}$, where M and are the first- and second-order parts, respectively. We assume familiarity with basic notions of model theory and reverse mathematics. The reader may refer to Chang and Keisler [7] and Simpson [63] for background on model theory and on reverse mathematics, respectively.

In subsection §3.1.2 we give a quick account of how concepts from model theory are formalized in the language of second-order arithmetic. In subsection §3.1.3 we give some useful characterizations of the principles ACA_0, WKL_0, $\mathsf{I}\Sigma^0_2$, and $\mathsf{B}\Sigma^0_2$. Section §3.2 presents our main results, organized thematically into smaller subsections §3.2.1 through §3.2.5. Although §3.2 includes some proofs, the majority are too long and are instead deferred variously to sections §3 through §7. Section §3 begins with an introductory part summarizing a method introduced in Chapter 2, and then moves on to an 'Applications' subsection §3.1. Each section among §4 through §6 describes a new construction or class of constructions, and is divided into four parts: first, an unnumbered introductory part which describes the construction and its goals in inexact terms; second, a 'Construction' subsection giving the details; third, a 'Verification' subsection where we check basic properties (such as completeness and consistency of a theory); and fourth, an 'Applications' subsection where the construction is used to prove theorems from section §3.2. Section §7 follows this pattern but has two 'Applications' subsections to accommodate some small twists on the construction.

3.1.2 Formalizing model theory

A *language* L is a sequence of relation symbols and function symbols together with their arities. An *L-formula* and *L-sentence* are defined as usual. Rules for deduction and a sequent calculus can be formalized—see Simpson [63, section II.8]. An *L-theory* is a set of L-sentences. A *consistent L-theory* is one not entailing the contradiction $\neg x = x$. A *complete L-theory* is an L-theory containing either ϕ or $\neg\phi$ for every L-sentence ϕ. An *L-structure* is a sequence of elements $a_0, a_1, \ldots$ (its *domain*) together with a complete consistent $L \cup \{a_0, \ldots\}$-theory (its *elementary diagram*) containing the set $\{a_i \neq a_j : i \neq j\}$. When no confusion arises we omit L and talk simply of *formulas*, *theories*, etc.

Fix a language L and an L-theory T. A *model of T* is a structure whose elementary diagram contains T. T is *satisfiable* if it has a model. An *n-type of T* is a set $p(x_0, \ldots, x_{n-1})$ of L-formulas with variables in $\{x_0, \ldots, x_{n-1}\}$ such that $\{\phi(c_{i_0}, \ldots, c_{i_{k-1}}) : \phi(x_{i_0}, \ldots, x_{i_{k-1}}) \text{ is in} p(x_0, \ldots, x_{n-1})\}$ is a complete consistent $L \cup \{c_0, \ldots, c_{n-1}\}$-theory, where $c_0, \ldots, c_{n-1}$ are new constant symbols. We often shorten $p(x_0, \ldots, x_{n-1})$ to p. We also often drop the n and refer to p as simply a *type*.

An n-type p of T is *principal* if there is a formula $\phi \in p$ such that p is the only n-type of T containing ϕ. Otherwise, p is *nonprincipal*. If $\bar{a}$ is a sequence of n elements of a model $\mathcal{A}$ of T, then $\mathsf{tp}^{\mathcal{A}}(\bar{a})$ is defined as the set of all n-ary formulas such that $\mathcal{A} \models \phi(\bar{a})$. Note that $\mathsf{tp}^{\mathcal{A}}(\bar{a})$ is an n-type. If p is a type and $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = p$ for some $\bar{a}$, we say that $\mathcal{A}$ *realizes* p and that $p(\bar{a})$ *holds*. Otherwise, $\mathcal{A}$ *omits* p.

We now consider some model-theoretic notions that do not admit a unique formulation in second-order arithmetic—or rather, they have several formulations which classically are considered equivalent and interchangeable, but which are not provably equivalent in RCA_0.

Definition 3.1.1. Fix a complete theory T and a model $\mathcal{A}$ of T.

1. $\mathcal{A}$ is *atomic* if every type realized by $\mathcal{A}$ is principal.
2. $\mathcal{A}$ is *prime* if it embeds elementarily into every model of T.
3. $\mathcal{A}$ is *1-point homogeneous* if for every pair $\bar{a}, \bar{b}$ of tuples such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{b})$ and every element u, there is an element v such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}^\frown u) = \mathsf{tp}^{\mathcal{A}}(\bar{b}^\frown v)$. (Here '$^\frown$' denotes concatenation of tuples.)
4. $\mathcal{A}$ is *1-homogeneous* if for every pair $\bar{a}, \bar{b}$ of tuples such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{b})$ and every tuple $\bar{u}$, there is a tuple $\bar{v}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}^\frown \bar{u}) = \mathsf{tp}^{\mathcal{A}}(\bar{b}^\frown \bar{v})$.
5. $\mathcal{A}$ is *strongly 1-homogeneous* if for every pair $\bar{a}, \bar{b}$ of tuples such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{b})$, there is an automorphism of $\mathcal{A}$ which maps each entry of $\bar{a}$ to the corresponding entry of $\bar{b}$.

6. $\mathcal{A}$ is *homogeneous* if for every finite sequence of tuples $\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}_i) = \mathsf{tp}^{\mathcal{A}}(\bar{b}_i)$ for all $i < n$, and every sequence of tuples $\bar{u}_0, \ldots, \bar{u}_{n-1}$, there is a second sequence $\bar{v}_0, \ldots, \bar{v}_{n-1}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}_i \frown \bar{u}_i) = \mathsf{tp}^{\mathcal{A}}(\bar{b}_i \frown \bar{v}_i)$ for all $i < n$.

7. $\mathcal{A}$ is *saturated* if, for every tuple $\bar{a}$ from its domain, the model $(\mathcal{A}, \bar{a})$ realizes every type of the theory $\mathsf{tp}^{\mathcal{A}}(\bar{a})$.

8. $\mathcal{A}$ is *universal* if every model of T embeds elementarily into $\mathcal{A}$.

Items 1 and 2 are classically equivalent; as are 3, 4, 5, and 6. Furthermore, 7 classically implies 8. None of these equivalences or implications is provable from RCA_0; their precise strengths are explored variously in Hirschfeldt, Shore, and Slaman [30], Hirschfeldt, Lange, and Shore [28], and Harris [25].

3.1.3 The basics of WKL_0, ACA_0, $\mathsf{I}\Sigma^0_2$, and $\mathsf{B}\Sigma^0_2$

Each of our new results involves one of the following well-known axioms: Weak König's Lemma, the Arithmetic Comprehension Axiom, Σ^0_2 induction, and Σ^0_2 bounding. When combined with RCA_0, these form the axiom systems WKL_0, ACA_0, $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$, and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, respectively. In this subsection we define and give some alternate characterizations of each of these principles. The uninterested reader may skip it and refer back as needed.

Definition 3.1.2. The Arithmetic Comprehension Axiom is axiom scheme: For each arithmetical formula $\phi(x)$ is an arithmetical formula in the language of second-order arithmetic with a free first-order variable x and an arbitrary set as a parameter, there is a set C such that $\phi(x) \leftrightarrow x \in C$. We use ACA_0 to denote RCA_0+Arithmetic Comprehension Axiom.

Simpson [63] and others have compiled impressive lists of natural statements equivalent to ACA_0 over RCA_0. We content ourselves with just the computability-theoretic principle given as item (ii) of the following lemma.

Lemma 3.1.3. *The following are equivalent over* RCA_0*:*

(i) ACA_0

(ii) *For every set Z, there is a second set K^Z consisting of all e such that $\Phi_e(e)$ converges, where Φ_e is the e-th Turing machine.*

Proof. See Simpson [63, Ex. VIII.1.12]. □

The set K^Z is called the *Turing jump of Z*. Lemma 3.1.3 is commonly used for proving ACA_0 from some other principle. It reduces the task from showing the existence of infinitely many sets—one for each arithmetical formula with set parameters—to that of showing the existence of a single, well-understood set K^Z, with Z ranging over .

Definition 3.1.4. Weak König's Lemma is the statement: Every infinite binary tree has an infinite path. We use WKL_0 to denote RCA_0+Weak König's Lemma.

WKL_0 is strong enough to carry out certain compactness arguments that do not work in RCA_0 alone. In fact, WKL_0 is equivalent over RCA_0 to many well-known facts, among them numerous compactness theorems. The following lemma lists a few useful characterisations of WKL_0; much longer lists can be found in Simpson [63].

Lemma 3.1.5. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) *The Compactness Theorem for first-order logic: If T is a set of first-order sentences and every finite subset of T is satisfiable, then T is satisfiable.*

(iii) *The* Σ^0_1 *separation principle: If* $\phi(x,s)$ *and* $\psi(x,s)$ *are quantifier-free formulas in the language of second-order arithmetic with set parameters, and* $(\forall x \forall s \forall t)[\neg\phi(x,s) \vee \neg\phi(x,t)]$*, then there is a set* C *such that* $(\exists s)\phi(x,s)$ *implies* $x \in C$*, and* $(\exists s)\psi(x,s)$ *implies* $x \notin C$*.*

Proof. For (i $\leftrightarrow$ ii), see Simpson [63, Thm IV.3.3]. For (i $\leftrightarrow$ iii), see [63, Lem IV.4.4]. □

We make use of all three equivalent statements (i), (ii), (iii) in this paper: We use Weak König's Lemma in its original form in §3.3, in the form of the Σ^0_1 separation principle in §3.6 and §3.7, and the first-order Compactness Theorem throughout. We now introduce a few definitions that make the Σ^0_1 separation principle easier to work with.

Definition 3.1.6. 1. A *disjoint* Σ^0_1 *pair* is a sequence $\langle U_s, V_s \rangle_{s \in M}$ of pairs U_s, V_s with the following properties:

- Each U_s and V_s is finite, with $\max(U_s \cup V_s) < s$.
- $U_s \cap V_s = \emptyset$ for every s.
- $U_s \subseteq U_{s+1}$ and $V_s \subseteq V_{s+1}$ for every s.

2. Given a disjoint Σ^0_1 pair $\langle U_s, V_s \rangle_s$, a set $C \subseteq M$ is called a *separating set for* $\langle U_s, V_s \rangle_s$ if, for every s, we have $U_s \subseteq C \subseteq (M - V_s)$. If no such C exists, then $\langle U_s, V_s \rangle_s$ is called an *inseparable* Σ^0_1 *pair*.

The Σ^0_1 separation principle can be phrased in these terms:

Theorem 3.1.7. *The following are equivalent over* RCA_0*:*

(i) The Σ^0_1 *separation principle*

(ii) The statement, 'There is no inseparable Σ^0_1 *pair'.*

We now turn to induction and bounding principles.

Definition 3.1.8. The Σ^0_2 *induction scheme* is the axiom scheme: For each Σ^0_2 formula $\phi(n)$ in the language of second-order arithmetic with one free first-order variable n and an arbitrary set as a parameter, the formula $(\phi(0) \wedge (\forall n)\phi(n) \rightarrow \phi(n+1)) \rightarrow (\forall n)\phi(n)$ holds. We use $\mathsf{I}\Sigma^0_2$ to represent the Σ^0_2 induction scheme.

Note that, because set parameters are allowed, this $\mathsf{I}\Sigma^0_2$ is not the same as the $\mathsf{I}\Sigma_2$ studied in the setting of first-order Peano arithmetic. Note also that Simpson [63] uses the notation Σ^0_2-IND where we would write $\mathsf{I}\Sigma^0_2$. Like the other principles under consideration, $\mathsf{I}\Sigma^0_2$ can be phrased in a number of equivalent ways:

Lemma 3.1.9. *The following are equivalent over* RCA_0*:*

(i) $\mathsf{I}\Sigma^0_2$

(ii) $\mathsf{L}\Pi^0_2$*: If ψ is a Π^0_2 formula, and there is an n such that $\psi(n)$ holds, then there is a least such n.*

(iii) *If $\langle D_1 \subseteq D_2 \subseteq \ldots\rangle$ is an increasing sequence of sets (coded as a single set) such that, for each n, D_n finite implies that D_{n+1} is finite, then either D_n is finite for all n, or D_n is infinite for all n.*

(iv) *If $\langle D_1 \subseteq D_2 \subseteq \ldots\rangle$ is an increasing sequence of sets (coded as a single set) such that, for each n, D_n finite implies that D_{2n} is finite, then either D_n is finite for all n, or D_n is infinite for all n.*

Proof. The equivalence (i $\leftrightarrow$ ii) is well-known; a proof can be adapted from the first-order case, found in Hajek and Pudlak [23]. The directions (i $\rightarrow$ iii) and (iii $\rightarrow$ iv) are immediate.

Now we show that (iv) implies (ii). Suppose that ψ is a Π^0_2 formula given by $\psi(i) \Leftrightarrow (\forall x \exists y)\phi(i,x,y)$, where ϕ is Σ^0_0. For each $n \geq 1$, define

$$D_n = \{\langle i,s,t\rangle : \ i < \log_2 n \text{ and } t \text{ is least s.t. } (\forall x < s)(\exists y < t)\phi(i,x,y)\}.$$

These D_n form an increasing chain of sets, D_1 is empty, and, whenever D_n is finite and $\psi(\lfloor \log_2 n \rfloor)$ does not hold, we have D_{2n} finite as well; on the other hand, if $\psi(\lfloor \log_2 n \rfloor)$ holds, then D_{2n} is infinite. Now suppose that there is no least i satisfying ψ. Then (iv) implies D_n is finite for all n, and, in particular, that no i satisfies ψ is empty. $\square$

Although they are relatively complicated to state, their use of sets in place of formulas makes (iii) and (iv) easier to use for some constructions when we work in a model of $\neg \mathsf{I}\Sigma^0_2$—see, for example, the constructions in §3.7. We also use the original formulation (i) of $\mathsf{I}\Sigma^0_2$ several times in §3.2.2. We make no further mention of (ii).

Definition 3.1.10. The Σ^0_2 *bounding principle* is the axiom scheme: For each Π^0_1 formula $\phi(i, x)$ in the language of second-order arithmetic with two free first-order variables i, x and an arbitrary set as a parameter, the formula

$$((\forall i < n)(\exists x)\phi(i, x)) \to (\exists x_0)(\forall i < n)(\exists x < x_0)\phi(i, x)$$

holds. We use $\mathsf{B}\Sigma^0_2$ to represent the Σ^0_2 bounding principle.

As with Σ^0_2 induction, we hasten to point out that $\mathsf{B}\Sigma^0_2$ is not the same as the principle $\mathsf{B}\Sigma_2$ studied in first-order arithmetic. We also point out one alternate characterization:

Lemma 3.1.11. *The following are equivalent over* RCA_0*:*

(i) $\mathsf{B}\Sigma^0_2$

(ii) *For each Π^0_1 formula $\psi(i, x)$ with an arbitrary set as a parameter,*

$$((\forall i < n)(\exists x)\psi(i, x)) \to (\exists \text{ a tuple } \langle x_0, \ldots, x_{n-1}\rangle)\psi(0, x_0) \wedge \cdots \wedge \psi(n-1, x_{n-1})$$

holds.

3.2 Main Results

Our results are organized into five subsections. The first two deal with existence theorems for homogeneous and saturated models, respectively; the third, with type amalgamation properties and the relations between them; the fourth, with elementary embeddings and prime and universal models; and the fifth, with the strength of the existence theorem for indiscernibles.

3.2.1 Existence theorems for homogeneous models

Consider the following well-known fact of classical model theory.

Theorem 3.2.1 (Weak homogeneous model existence theorem. Classical)**.** *If T is a complete consistent countable theory, then T has a countable homogeneous model.*

The word *Weak* is meant to distinguish this theorem from a stronger version which does not require T to be complete. What is the strength of Theorem 3.2.1 over RCA_0? In Definition 3.1.1, we gave a number of different formalizations of the term *homogeneous* in the language of second-order arithmetic. On the face of it it looks as though the corresponding versions of the existence theorem may have wildly different strengths. Lange in her thesis showed the following:

Theorem 3.2.2 (Lange [40])**.** $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow$ *Every complete consistent theory has a 1-point homogeneous model.*

In fact, three of the four versions of homogeneity from Definition 3.1.1 give a statement of equivalent strength:

Theorem 3.2.3. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) *Every complete consistent theory has a 1-point homogeneous model.*

(iii) *Every complete consistent theory has a 1-homogeneous model.*

(iv) *Every complete consistent theory has a strongly 1-homogeneous model.*

A proof of Theorem 3.2.3 is implicit in Lange's proof of Theorem 3.2.2. We give an alternate proof and some extensions of (i $\leftrightarrow$ iv) in §3.3. Our first new result extends (i $\leftrightarrow$ iv) by introducing restrictions on the types of T:

Theorem 3.2.4. $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow$ *Every complete consistent theory with only principal types has a strongly 1-homogeneous model.*

Proof. The $\rightarrow$ direction is immediate from Theorem 3.2.3. The $\leftarrow$ direction is proved as Proposition 3.5.6 below. □

On the other hand, Hirschfeldt, Lange, and Shore [28] have shown that if one first specifies the type spectrum of the required model, following Goncharov [21] and Peretyatkin [54], one ends up with a large number of nonequivalent statements.[1] As well, we do not know much about the strength of Theorem 3.2.1 when we use the fourth, remaining formalization of homogeneity from Definition 3.1.16, except that, using Theorem 3.2.4 and results from [28], it is provable from $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Question 3.2.5. What is the strength over RCA_0 of the statement, 'Every complete consistent theory has a homogeneous model in the sense of Definition 3.1.1'? Is it equivalent to $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

[1] For example, they find one equivalent to $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ over RCA_0; one provable in $\Pi^0_1\mathsf{GA}$ but not in RCA_0; and one provable in $\Pi^0_1\mathsf{GA}$ and equivalent to $\mathsf{I}\Sigma^0_2$ over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

3.2.2 Existence theorems for saturated models

We have already given a definition of *saturated* in second-order arithmetic as part of Definition 3.1.1. We begin this subsection with a second, weaker notion.

Definition 3.2.6. Let T be a complete theory, and $\mathcal{A}$ a model of T. We say that $\mathcal{A}$ is $\emptyset$*-saturated* if it realizes every type of T.

The following characterization of saturated models, well-known in the classical setting, also holds in RCA_0. It will be helpful in the work that follows.

Lemma 3.2.7. *Let T be a complete theory, and $\mathcal{A}$ a model of T. Then $\mathcal{A}$ is saturated if and only if $\mathcal{A}$ is both $\emptyset$-saturated and 1-homogeneous.*

Proof. First we show the 'only if' direction. Suppose that $\mathcal{A}$ is saturated. It is immediate from the definition that $\mathcal{A}$ is $\emptyset$-saturated as well. To see that $\mathcal{A}$ is 1-homogeneous, choose any three tuples $\bar{a}, \bar{b}, \bar{u}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{u})$. Let $p = \mathsf{tp}^{(\mathcal{A},\bar{a})}(\bar{b})$ be the type of $\bar{b}$ over the enriched structure $(\mathcal{A}, \bar{a})$; since $\mathcal{A}$ is saturated, there is a tuple $\bar{v}$ such that $\mathsf{tp}^{(\mathcal{A},\bar{u})}(\bar{v}) = p$. Hence $\mathsf{tp}^{\mathcal{A}}(\bar{a} ^\frown \bar{b}) = \mathsf{tp}^{\mathcal{A}}(\bar{u} ^\frown \bar{v})$, so $\mathcal{A}$ is 1-homogeneous.

Next we deal with the 'if' direction. Suppose that $\mathcal{A}$ is $\emptyset$-saturated and 1-homogeneous. Let $\bar{a}$ be any tuple, and let $p(\bar{y})$ be any type of the theory $\mathsf{tp}^{\mathcal{A}}(\bar{a})$. Replace the constants $\bar{a}$ in p with new variables $\bar{x}$ to get a type $p'(\bar{x} ^\frown \bar{y})$ of T. This p' is realized by some tuple $\bar{u} ^\frown \bar{v}$ from $\mathcal{A}$, with $\mathsf{tp}^{\mathcal{A}}(\bar{u}) = \mathsf{tp}^{\mathcal{A}}(\bar{a})$. Hence, by 1-homogeneity, there is a tuple $\bar{b}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a} ^\frown \bar{b}) = p'$, as desired. $\square$

Now consider the following well-known theorem.

Theorem 3.2.8 (Weak saturated model existence theorem. Classical)**.** *If T is a complete consistent theory with only countably many types, then T has a countable saturated model.*

As we did for the homogeneous case at the start of §3.2.1, we ask for the reverse-mathematical strength of Theorem 3.2.8. And as in the homogeneous case, we must begin by formalizing the statement in second-order arithmetic. We have already settled on a suitable notion of saturation; our next worry is the notion of *countably many types*.

Definition 3.2.9. Fix a complete consistent theory T.

1. A *sequence of types* of T is a coded sequence $X = \langle p_0, p_1, \ldots \rangle$ such that each p_i is a type of T. X is a *sequence of all types of* T if every type of T is equal to some p_i.
2. We say T *has countably many types* if it has a sequence of all types.

Even given this definition, there are a number of different ways to formalize and analyze Theorem 3.2.8. We begin with the most basic:

Theorem 3.2.10. *The following are equivalent over* RCA_0*.*

(i) WKL_0

(ii) *Every complete consistent theory with countably many types has a saturated model.*

Proof. The (i $\to$ ii) direction follows from Corollary 3.3.4 below. The (ii $\to$ i) direction is immediate from Proposition 3.6.5 below. □

The proof of the (ii $\to$ i) direction works by assuming $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, and constructing a complete consistent theory T with two types p and q that can never be realized in the same model. We can rule out this obstruction by requiring that a theory's types have one of the following amalgamation properties.

Definition 3.2.11. Fix a complete consistent theory T and a sequence $X = \langle q_0, \ldots \rangle$ of types of T.

1. We say X has the *pairwise full amalgamation property* if, for every type $p(\bar{x})$ and every pair $q_i(\bar{x}, \bar{y})$, $q_j(\bar{x}, \bar{z})$ of types in X extending p, there is a type $r(\bar{x}, \bar{y}, \bar{z})$ in X extending both q_i and q_j.

2. We say X has the *finite full amalgamation property* if, for every type $p(\bar{x})$ and every tuple $\langle i_0, \ldots, i_{n-1} \rangle$ of indices such that $q_{i_k}(\bar{x}, \bar{y}_k)$ extends p for each $k < n$, there is a type $r(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1})$ in X extending each q_{i_k}.

Proposition 3.2.12. *Suppose T is a complete theory with a saturated model. Then T has countably many types, and every enumeration of all types of T has the pairwise full amalgamation property.*

Proof. Fix a saturated model $\mathcal{A}$ of T. We can enumerate the tuples $\langle \bar{a}_k \rangle_{k \in M}$ in $\mathcal{A}$, and hence enumerate the types $\langle p_k \rangle_{k \in M}$ realized in $\mathcal{A}$ by $p_k =$ the type realized by $\bar{a}_k$. Call this enumeration X. Clearly X is an enumeration of all types of T, so by definition, T has countably many types.

Now let Y be any enumeration of all types of T. To see that Y has the pairwise full amalgamation property, consider any type $p(\bar{x})$ and any two types $q_0(\bar{x}, \bar{y}), q_1(\bar{x}, \bar{z})$ of T extending p. Since $\mathcal{A}$ is $\emptyset$-saturated, it realizes q_0 and q_1, say with tuples $\bar{a}^\frown\bar{b}$ and $\bar{u}^\frown\bar{v}$, respectively, where $|\bar{a}| = |\bar{u}| = |\bar{x}|$ and $|\bar{b}| = |\bar{v}| = |\bar{y}|$. Since $\bar{a}$ and $\bar{u}$ realize the same type p, and since $\mathcal{A}$ is 1-homogeneous by Lemma 3.2.7, there is a tuple $\bar{c}$ such that $\mathsf{tp}(\bar{a}, \bar{c}) = \mathsf{tp}(\bar{u}, \bar{v}) = q_1$. Let $r(\bar{x}, \bar{y}, \bar{z}) = \mathsf{tp}(\bar{a}, \bar{b}, \bar{c})$. Then r extends $q_0(\bar{x}, \bar{y}) \cup q_1(\bar{x}, \bar{z})$. Hence we conclude that every enumeration of all types of T has the pairwise full amalgamation property. $\square$

In classical model theory, the converse of Proposition 3.2.12 is usually proved by a compactness argument; in the present setting, such a proof requires WKL_0. In effective model theory, the converse is instead usually proved, following Millar [46, 45] and Morley [50], by a finite injury argument. This requires $\mathsf{I}\Sigma^0_2$. Hence we arrive at the following:

Proposition 3.2.13. $\mathsf{RCA}_0 + (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2) \vdash$ *Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model.*

Remarkably, if we include $\mathsf{B}\Sigma^0_2$ as an assumption, Proposition 3.2.13 admits a reversal.

Theorem 3.2.14. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2) \leftrightarrow$ *Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model.*

Proof. The $\rightarrow$ direction is a weakening of Proposition 3.2.13. The $\leftarrow$ direction is proved as Proposition 3.7.17 below. □

An obvious question is whether $\mathsf{B}\Sigma^0_2$ can be dropped in the statement of Theorem 3.2.14. We answer this question in the negative in Corollary 3.2.19 below. Our answer uses recent results about the combinatorial principle $\Pi^0_1\mathsf{GA}$, which states, roughly: For every sequence $\mathcal{D}$ of dense uniformly Π^0_1 subsets of $2^{<\mathbb{N}}$, there is a sequence $\sigma_0, \sigma_1, \ldots \in 2^{<\mathbb{N}}$ whose pointwise limit exists and is $\mathcal{D}$-generic. (Refer to [28] for a rigorous definition.) In terms of reverse-mathematical strength, this principle falls somewhere between $\mathsf{I}\Sigma^0_1$ and $\mathsf{I}\Sigma^0_2$, and is incomparable with $\mathsf{B}\Sigma^0_2$.

Theorem 3.2.15 (Hirschfeldt, Lange, Shore [28])**.** (i) $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \nvdash \Pi^0_1\mathsf{GA}$

(ii) $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 \vdash \Pi^0_1\mathsf{GA}$

(iii) $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \Pi^0_1\mathsf{GA} \vdash \mathsf{I}\Sigma^0_2$

A further result in [30] is that the principle $\Pi^0_1\mathsf{G}$, which is stronger than $\Pi^0_1\mathsf{GA}$, has a certain conservation property over RCA_0. From this we deduce:

Theorem 3.2.16. $\mathsf{RCA}_0 + \Pi^0_1\mathsf{GA} \nvdash \mathsf{WKL}_0 \vee \mathsf{B}\Sigma^0_2$

Proof. Immediate from the observation in [30, section 4] that $\Pi^0_1\mathsf{G}$ is restricted Π^1_2 conservative over RCA_0, and from the fact that $\Pi^0_1\mathsf{G}$ implies $\Pi^0_1\mathsf{GA}$. (Both $\Pi^0_1\mathsf{G}$ and *restricted* Π^1_2 *conservative* are defined in [30].) □

As mentioned above, the converse of Proposition 3.2.12 can be proved using $\mathsf{I}\Sigma^0_2$. In fact, the weaker axiom $\Pi^0_1\mathsf{GA}$ is already enough to prove a similar theorem:

Theorem 3.2.17 (Hirschfeldt, Lange, and Shore [28])**.** $\mathsf{RCA}_0 + \Pi^0_1\mathsf{GA} \vdash$ *If T is a complete consistent theory and X is a sequence of types with the pairwise full amalgamation property, then T has a 1-homogeneous model which realizes exactly the types in X.*

Hence we derive:

Corollary 3.2.18. $\mathsf{RCA}_0 + (\mathsf{WKL}_0 \vee \Pi^0_1\mathsf{GA}) \vdash$ *Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model.*

Proof. WKL_0 proves the given statement by Theorem 3.2.10. $\Pi^0_1\mathsf{GA}$ proves the statement by Proposition 3.2.17 and Lemma 3.2.7. □

This allows us to prove that the assumption of $\mathsf{B}\Sigma^0_2$ cannot be dropped from the statement of Theorem 3.2.14:

Corollary 3.2.19. $\mathsf{RCA}_0 \nvdash$ *(Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model)*$\rightarrow (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2)$.

Proof. By Theorem 3.2.16 we may fix a model $(M,)$ of $\mathsf{RCA}_0 + \Pi^0_1\mathsf{GA} + \neg\mathsf{B}\Sigma^0_2 + \neg\mathsf{WKL}_0$. Then by Theorem 3.2.17 there is a theory $T \in$ as in the corollary statement, but $(M,)$ is neither a model of WKL_0 (by assumption) nor of $\mathsf{I}\Sigma^0_2$ (since $\mathsf{I}\Sigma^0_2$ implies $\mathsf{B}\Sigma^0_2$). □

On the other hand, these results suggest the following, weaker question, to which we do not know the answer.

Question 3.2.20. Is the statement, 'Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model' equivalent to $\mathsf{WKL}_0 \vee \Pi^0_1\mathsf{GA}$ over RCA_0?

3.2.3 Type amalgamation, WKL_0, and induction

Recall from Definition 3.2.11 the *pairwise full amalgamation property* and the *finite full amalgamation property.* We now list four more properties in the same family.

Definition 3.2.21 (Hirschfeldt, Lange, and Shore [28])**.** Fix a complete consistent theory T and a sequence $X = \langle q_0, \ldots \rangle$ of types of T.

1. We say X has the *1-point full amalgamation property* if for every n-type $p(\bar{x})$ in X and every pair of $(n+1)$-types $q_0(\bar{x}, y)$, $q_1(\bar{x}, z)$ in X extending p, there is an $(n+2)$-type $r(\bar{x}, y, z)$ in X extending both q_0 and q_1.
2. We say X has the *1-point free amalgamation property* if for every n-type $p(\bar{x})$ in X and every 1-type $q(y)$ in X, there is an $(n+2)$-type $r(\bar{x}, y)$ in X extending both p and q.
3. We say X has the *pairwise free amalgamation property* if for every pair $p(\bar{x})$, $q(\bar{y})$ of types in X, there is a type $r(\bar{x}, \bar{y}, \bar{z})$ in X extending both q_i and q_j.
4. We say X has the *finite free amalgamation property* if given any tuple $\langle i_0, \ldots, i_{n-1} \rangle$ of indices such that the variables of $q_{i_k}(\bar{y}_k)$ are pairwise disjoint, there is a type $r(\bar{y}_0, \ldots, \bar{y}_{n-1})$ in X extending each q_{i_k}.

These amalgamation properties are based on those used by Goncharov [21] and Peretyatkin [54] in studying homogeneous models in effective mathematics. We are interested in the special case where X is the sequence of all types of T; the situation for more general X is explored in [28]. We introduce six predicates which take as their argument a set X, and which abbreviate the six kinds of amalgamation property. The following serves as a prototype:

- $\mathsf{1PT\ FREE}(X) \Leftrightarrow X$ is a sequence of all types of a complete consistent theory T with the 1-point free amalgamation property.

The predicates $\mathsf{1PT\ FULL}(X)$, $\mathsf{PW\ FREE}(X)$, $\mathsf{PW\ FULL}(X)$, $\mathsf{FIN\ FREE}(X)$, and $\mathsf{FIN\ FULL}(X)$ are defined analogously for the 1-point full, pairwise free, pairwise full, finite free, and finite full amalgamation properties, respectively.

Theorem 3.2.22. (i) $\mathsf{WKL_0} \vdash (\forall X)\mathsf{1PT\ FREE}(X) \to \mathsf{FIN\ FULL}(X)$.

(ii) $\mathsf{RCA_0} + \mathsf{I\Sigma^0_2} \vdash (\forall X)\mathsf{1PT\ FREE}(X) \to \mathsf{FIN\ FREE}(X)$.

(iii) $\mathsf{RCA_0} + \mathsf{I\Sigma^0_2} \vdash (\forall X)\mathsf{1PT\ FULL}(X) \to \mathsf{FIN\ FULL}(X)$.

Proof. Item (i) is immediate by the Compactness Theorem. (And in fact, this proof does not require the 1-point free amalgamation property as an assumption.) Items (ii) and (iii) are each proved by a straightforward induction. □

Theorem 3.2.23.

(i) $\mathsf{RCA_0} \vdash (\forall X)[\mathsf{1PT\ FULL}(X) \to \mathsf{PW\ FREE}(X)] \to \mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}$.

(ii) $\mathsf{RCA_0} \vdash (\forall X)[\mathsf{PW\ FULL}(X) \to \mathsf{FIN\ FREE}(X)] \to \mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}$.

(iii) $\mathsf{RCA_0} \vdash (\forall X)[\mathsf{FIN\ FREE}(X) \to \mathsf{1PT\ FULL}(X)] \to \mathsf{WKL_0}$.

Proof. Item (i) is proved as Proposition 3.7.11 below. Item (ii) is proved as Proposition 3.7.15. Item (iii) is proved as Proposition 3.6.4. □

	1PT FREE	PW FREE	FIN FREE	1PT FULL	PW FULL	FIN FULL
1PT FREE		$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	WKL_0	WKL_0	WKL_0
PW FREE			$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	WKL_0	WKL_0	WKL_0
FIN FREE				WKL_0	WKL_0	WKL_0
1PT FULL		$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$		$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$
PW FULL			$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$			$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$
FIN FULL						

Table 3.1: Implications for Theorem 3.2.24.

Theorem 3.2.24. *The table in Figure 3.1 has the following property. If a principle P is listed in the row corresponding to an amalgamation property A and the column corresponding to an amalgamation property B, then*

$$\mathsf{RCA}_0 \vdash P \leftrightarrow (\forall X)[A(X) \to B(X)].$$

If the cell in row A and column B is greyed out, then $\mathsf{RCA}_0 \vdash (\forall X)[A(X) \to B(X)]$ immediately from the definitions.

Proof. For every cell in row A and column B which is not greyed out, the implication $A \to B$ is weaker than one or more implications mentioned in Theorem 3.2.22 and stronger than one mentioned in Theorem 3.2.23. It is straightforward in each case to compare the facts from these two theorems and arrive at the promised result. $\square$

3.2.4 Elementary embeddings and universal models

Here we consider certain existence theorems for elementary embeddings between models, and for models which have elementary embeddings between them.

Theorem 3.2.25. WKL_0 *proves the following. Suppose T is a complete theory, and $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ is a countable sequence of models of T. Then there is a model $\mathcal{B}$ of T such that each $\mathcal{A}_j$ embeds elementarily into $\mathcal{B}$.*

Proof. See §3.3.1 below. $\square$

Recall from Definition 3.1.1 the notion of a *universal* model. Theorem 3.2.25 has an immediate corollary in terms of universal models:

Corollary 3.2.26. $\mathsf{WKL}_0 \vdash$ *If T is a complete theory and there is a listing $\langle \mathcal{A}_0, \ldots \rangle$ of all models of T up to isomorphism, then T has a universal model.*

We can also guarantee the existence of a universal model by looking at the number of n-types:

Theorem 3.2.27. WKL_0 *proves the following. Suppose that T is a complete theory, and $f : M \to M$ is a function such that $f(n)$ is greater than the number of n-types of T for all n. Then T has a universal model.*

Proof. See §3.3.1. $\square$

It is easy to see that the relation of elementary embeddability is reflexive and transitive—that is, it forms a preorder on models of T. Our next result shows that the conjunction $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ is equivalent over RCA_0 to a peculiar but natural statement about this preorder. A closely-related statement, weaker on its face but also equivalent to $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$, can be found in Chapter 2.

Theorem 3.2.28. *The following are equivalent over* RCA_0*:*

(i) $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$

(ii) *If T is a theory which has infinitely many n-types for some n, then any partial order can be embedded into the preorder of models under elementary embedding.*

Proof. The (i $\to$ ii) direction is proved as Proposition 3.3.6 below. For the (ii $\to$ i) direction, we prove the contrapositive. Suppose that $(M,)$ is a model of $\mathsf{RCA}_0 +$

$\neg(\mathsf{WKL}_0 + \neg\mathsf{ACA}_0)$. In other words, $(M,)$ is a model either of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, or of ACA_0. If it is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, then there is a complete consistent theory $T \in$ with infinitely many 1-types but only one model up to isomorphism. (Use the T constructed in §3.4 below.) Otherwise, if it is a model of ACA_0, a classical construction due to Ehrenfeucht can be carried out to obtain a complete consistent theory $T \in$ with infinitely many 1-types and exactly three models up to isomorphism. (This construction can be found in Chang and Keisler [7].) □

3.2.5 Indiscernibles

Here we list one more consequence of the constructions in this paper.

Definition 3.2.29. Fix a language L, a complete consistent L-theory T and a model $\mathcal{A}$ of T. We say a sequence $\langle a_0, a_1, \ldots\rangle$ of distinct elements of $\mathcal{A}$ is *a sequence of indiscernibles* if, for every strictly increasing tuple $\langle i_0, \ldots, i_{n-1}\rangle$ of numbers, and for every n-ary L-formula ϕ, we have

$$\mathcal{A} \models \phi(a_0, \ldots, a_{n-1}) \text{ if and only if } \mathcal{A} \models \phi(a_{i_0}, \ldots, a_{i_{n-1}}).$$

The classical existence theorem for indiscernibles is:

Theorem 3.2.30 (Classical)**.** *Every complete consistent countable theory has a countable model with a sequence of indiscernibles.*

Indiscernibles have been studied in recursive model theory by Kierstead and Remmel [35, 36]. Among their results is the following bound on the classical existence theorem's complexity:

Theorem 3.2.31 (Kierstead and Remmel [36])**.** *There is a decidable theory for which every decidable model has a sequence of indiscernibles, but no decidable model has a sequence of indiscernibles which is hyperarithmetic.*

When reasoning in second-order arithmetic, one might therefore suspect Theorem 3.2.30 to be strictly stronger than Δ^0_1-CA_0. However, we find that this is not the case. In fact, every decidable theory has a low model with a low sequence of indiscernibles.

Theorem 3.2.32. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) *Every complete consistent theory has a model with a sequence of indiscernibles.*

Proof. To see the (i $\to$ ii) direction, simply notice that WKL_0 is strong enough to carry out the classical proof of (ii) by way of the Compactness Theorem 3.1.5. The (ii $\to$ i) direction is proved as Proposition 3.4.7 below. $\square$

3.3 Models and embeddings from a tree of Henkin constructions

Fix a model $\langle M, \rangle$ of WKL_0, and suppose that $T \in$ is a complete theory. In this first unnumbered subsection, we describe a general method for representing models as trees of Henkin-style diagrams, and give an idea of how it is to be used. This replicates a similar description from Chapter 2. Afterwards, in §3.3.1, we use the method to prove several new results.

Definition 3.3.1. Fix a language L and a complete L-theory T.

- Let L' be the expanded language $L \cup \{c_0, c_1, \ldots\}$, where each c_i is a new constant symbol. Let $\langle \phi_s \rangle_s$ be a one-to-one enumeration of all L'-sentences. Define a $2^{<M}$-indexed sequence $\langle D_\sigma \rangle_{\sigma \in 2^{<M}}$ of sets of L'-sentences by

$$D_\sigma = \{\phi_s : s < |\sigma| \text{ and } \sigma(s) = 1\} \cup \{\neg\phi_s : s < |\sigma| \text{ and } \sigma(s) = 0\}.$$

Define a sequence $\langle W_s \rangle_{s\in M}$ of sets of L'-sentences by recursion:

$$
\begin{aligned}
W_0 &= \emptyset \\
W_{s+1} &= \begin{cases} W_s \cup \{\phi_s \to \psi(c_{2k+1})\} \text{ if } \phi_s \text{ is of the form } (\exists x)\psi(x), \\ \qquad\qquad \text{where } 2k+1 \text{ is the least odd index such} \\ \qquad\qquad \text{that } c_{2k+1} \text{ is not mentioned in } W_s \text{ or in } D_\sigma \\ \qquad\qquad \text{for any } \sigma \text{ of length } \leq s. \\ W_s \text{ if } \phi_s \text{ is not of this form.} \end{cases}
\end{aligned}
$$

The *tree of odd Henkin diagrams* is the tree $\mathcal{H} \subseteq 2^{<M}$ given by

$$\mathcal{H} = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \text{ is consistent}\}.$$

- Given an infinite path β in $\mathcal{H}$, let $D_\beta = \bigcup_{s\in M} D_{\beta\restriction s}$. Then D_β is a complete, consistent L'-theory. Define an equivalence relation E on the constants $\{c_0, c_1, \ldots\}$ by $c_i E c_j$ iff $D_\beta \vdash c_i = c_j$. Denote the E-equivalence class of c_i by $[c_i]_E$, and let $\langle b_0, b_1, \ldots \rangle$ be the one-to-one listing of all E-equivalence classes given by

$$b_m = [c_{i_m}]_E, \text{ where } i_m \text{ is least s.t. } c_{i_m} \notin b_k \text{ for all } k < m.$$

Let $\mathcal{B}$ be the L-structure such that, for any L-formula ϕ,

$$\mathcal{B} \models \phi(b_0, \ldots, b_{n-1}) \iff D_\beta \vdash \phi(c_{i_0}, \ldots, c_{i_{m-1}}).$$

Then $\mathcal{B}$ is a model of T. We say that $\mathcal{B}$ is the *Henkin model encoded by* β.

Our simplest constructions using $\mathcal{H}$ work as follows. Fix a theory T, let $\mathcal{H}$ be the tree of odd Henkin diagrams, and let P be a property desired of a model. We specify a subtree $\mathcal{H}^*$ of $\mathcal{H}$ by writing a set Φ_P of L'-sentences and letting $\mathcal{H}^*$ equal

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_P \text{ is consistent}\}.$$

Typically Φ_P is designed to ensure that any model encoded by a path of $\mathcal{H}^*$ has property P. We then show that $\mathcal{H}^*$ is an infinite tree. An appeal to Weak König's Lemma yields a model of T with the property P.

Some examples of such Φ_P are:

- A set Φ_H which ensures the model is strongly 1-homogeneous. (Proposition 3.3.2)
- A set Φ_S which ensures the model is $\emptyset$-saturated. (Proposition 3.3.3)
- The union $\Phi_H \cup \Phi_S$, which ensures the model is saturated using Lemma 3.2.7. (Corollary 3.3.4)
- Given a model $\mathcal{A}$ of T, a set $\Phi_{\mathcal{A}}$ which ensures that $\mathcal{A}$ embeds elementarily into the new model. (Theorem 3.2.25, proved below. A similar set appears in the proof of Proposition 3.3.6.)
- Given a model $\mathcal{A}$ of T, a set which ensures that either the new model embeds elementarily into $\mathcal{A}$, or ACA_0 holds. (Used in Chapter 2. A similar set is in the proof of Proposition 3.3.6.)

Sometimes we construct not one but a whole sequence $\langle \mathcal{B}_0, \mathcal{B}_1, \ldots \rangle$ of models with some property such as being pairwise non-isomorphic. We do this by considering the set $\{\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in \mathcal{H}^{<M} : \text{ each } \sigma_i \text{ has length } |\sigma_i| = n\}$ with the ordering $\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \prec \langle \tau_0, \ldots, \tau_{m-1} \rangle$ if $n \leq m$ and $\sigma_i \subseteq \tau_i$ for every $i < n$. Any path through this tree encodes a sequence $\langle \mathcal{B}_0, \ldots \rangle$ of models of T. For an example of this method, see the proof of Proposition 3.3.6 below.

3.3.1 Applications

Our first use of the tree of odd Henkin diagrams is to prove one direction of Theorem 3.2.3. An alternate proof of the same direction is implicit in Lange [40, Proof of Thm 4.3.1].

Proposition 3.3.2. $\mathsf{WKL}_0 \vdash$ *Every complete consistent theory has a strongly 1-homogeneous model.*

Proof. Let $(M,)$ be a model of WKL_0, and fix a complete consistent theory $T \in$. Define a sequence of finite sets $\Phi_{H,0} \subseteq \Phi_{H,1} \subseteq \cdots$ of L'-sentences by:

$$\begin{aligned}
\Phi_{H,0} &= \emptyset \\
\Phi_{H,s+1} &= \begin{cases} \Phi_{H,s} \cup \{\phi_s \to \psi(r ^\frown c_{2\langle \bar{p},q,\bar{r}\rangle})\} \text{ if } \phi_s \text{ is } \psi(\bar{p} ^\frown q) \wedge (\exists x)\psi(\bar{r} ^\frown x) \\ \qquad \text{with } \psi \text{ an } L\text{-formula, each } \bar{p}, q, \bar{r} \text{ taken from } \{c_i\}_{i \in M}, \\ \Phi_{H,s} \text{ if } \phi_s \text{ is not of this form.} \end{cases}
\end{aligned}$$

Let $\mathcal{H}^*$ be the subtree of $\mathcal{H}$ given by:

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{H,|\sigma|} \text{ is consistent}\}.$$

First we check that $\mathcal{H}^*$ is infinite. Fix a model $\mathcal{A}$ of T and a level s of $\mathcal{H}^*$. It is easy to see that there is some assignment of constants $c_i^{\mathcal{A}}$ such that $(\mathcal{A}, c_i^{\mathcal{A}}) \models T \cup W_{|\sigma|} \cup \Phi_{H,s}$, and furthermore that this $(\mathcal{A}, c_i^{\mathcal{A}})$ satisfies some D_σ with $|\sigma| = s$. It follows that σ is in $\mathcal{H}^*$. Apply Weak König's Lemma to get a path β in $\mathcal{H}^*$, and let $\mathcal{B}$ be the model encoded by β.

Now we argue that $\mathcal{B}$ is strongly 1-homogeneous. For this we use an effective back-and-forth argument; we show only the 'forth' direction, the 'back' direction being similar. Let $\bar{a}, \bar{b}$ be any pair of tuples such that $\mathsf{tp}^{\mathcal{B}}(\bar{a}) = \mathsf{tp}^{\mathcal{B}}(\bar{b})$. Let $\bar{d}, \bar{e}$ be tuples of constants in $\{c_0, \ldots\}$ such that $a_i = [d_i]_E$ and $b_i = [e_i]_E$ for each i. Let u be the least-indexed element of $\mathcal{A}$ not in $\bar{a}$, and let j be an index such that $u = [c_j]_E$. Now let $k = 2\langle \bar{d}, c_j, \bar{e}\rangle$ and $v = [c_k]_E$. Then $\mathsf{tp}^{\mathcal{B}}(\bar{a} ^\frown u) = \mathsf{tp}^{\mathcal{B}}(\bar{b} ^\frown \bar{v})$. Notice that the procedure for finding v from $\bar{a}, \bar{b}$, and u is effective, so that we can iterate the construction in a model of RCA_0. $\square$

Proposition 3.3.3. $\mathsf{WKL}_0 \vdash$ *Every complete consistent theory with countably many types has a $\emptyset$-saturated model.*

Proof. Let $(M,)$ be a model of WKL_0, and fix a complete consistent theory $T \in$ with an enumeration of all types $X = \langle p_0, \ldots \rangle$. Let $\langle \bar{d}_0, \ldots \rangle$ be a sequence of tuples of constants in $\{c_{2i} : i \in M\}$, where each $\bar{d}_j$ has the same arity as p_j and where no constant c_{2i} appears twice. Define a sequence of finite sets of L^*-sentences:

$$\Phi_{S,s} \quad = \quad \{\phi_t(\bar{d}_j) : j, t < s,\ \phi_t(\bar{x}) \in p_j(\bar{x})\}.$$

Let $\mathcal{H}^*$ be the subtree of $\mathcal{H}$ given by:

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{S,|\sigma|} \text{ is consistent}\}.$$

It can be checked as in the proof of Proposition 3.3.2 that $\mathcal{H}^*$ is infinite. Use Weak König's Lemma to get the model $\mathcal{B}$ encoded by some path β in $\mathcal{H}^*$. The resulting $\mathcal{B}$ is $\emptyset$-saturated, since each type p_j in X is realized by the tuple of elements interpreting $\bar{d}_j$. □

Corollary 3.3.4. $\mathsf{WKL}_0 \vdash$ *Every complete consistent theory with countably many types has a saturated model.*

Proof. Fix $(M,) \models \mathsf{WKL}_0$, a complete consistent theory $T \in$, and an enumeration $X = \langle p_0, \ldots \rangle$ of all types of T. Let $\Phi_{H,s}$ and $\Phi_{S,s}$ be sets of sentences as in the proofs of Proposition 3.3.2 and Proposition 3.3.3, respectively, except with each $\Phi_{H,s}$ using only every fourth constant c_{4i}, and each $\Phi_{S,s}$ using only every fourth c_{4i+2}. Let

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{H,|\sigma|} \cup \Phi_{S,|\sigma|} \text{ is consistent}\}.$$

Once again, we may check that $\mathcal{H}^*$ is an infinite tree. By Weak König's lemma, there is a model $\mathcal{B}$ of T encoded by some path β through $\mathcal{H}^*$. This $\mathcal{B}$ is both 1-homogeneous and $\emptyset$-saturated, and hence is saturated by Lemma 3.2.7. □

This method is also used to prove the results from section §3.2.4, which focus on the existence and nonexistence of elementary embeddings. We begin with the following:

Proof of Theorem 3.2.25. Let $(M,)$ be a model of WKL_0, and fix a complete consistent theory $T \in$ with a sequence of models $\langle \mathcal{A}_0, \ldots \rangle$. For simplicity, assume each $\mathcal{A}_i$ shares the same domain $A = \{a_0, a_1, \ldots\}$. For each $i \in M$, define a sequence of finite sets of L^*-sentences:

$$\Phi_{\mathcal{A}_i,s} = \{\phi_t(c_{2\langle i,k_0\rangle}, \ldots, c_{2\langle i,k_{n-1}\rangle}) : k_0, \ldots, k_{n-1}, t < s, \text{ and } \mathcal{A}_i \models \phi_t(a_{k_0}, \ldots, a_{k_{n-1}})\}.$$

Let $\mathcal{H}^*$ be the subtree of $\mathcal{H}$ given by:

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{\mathcal{A}_0,|\sigma|} \cup \cdots \cup \Phi_{\mathcal{A}_{|\sigma|},|\sigma|} \text{ is consistent}\}.$$

As in the proof of Proposition 3.3.2, we can check that $\mathcal{H}^*$ is an infinite tree. Use Weak König's Lemma to get a model $\mathcal{B}$ encoded by some path in $\mathcal{H}^*$. We claim that every $\mathcal{A}_i$ embeds elementarily into $\mathcal{B}$. To see this, it is enough to notice that whenever $\langle b_{j_0}, \ldots, b_{j_{n-1}}\rangle$ is the tuple of elements of $\mathcal{B}$ corresponding to the tuple of constants $\langle c_{2\langle i,0\rangle}, \ldots, c_{2\langle i,n-1\rangle}\rangle$, we have $\mathsf{tp}^{\mathcal{B}}(b_{j_0}, \ldots, b_{j_{n-1}}) = \mathsf{tp}^{\mathcal{A}}(a_{i,0}, \ldots, a_{i,n-1})$. $\square$

Next we wish to prove Theorem 3.2.27. The following will be helpful.

Lemma 3.3.5. WKL_0 *proves the following. If T is a complete theory and $\mathcal{A}, \mathcal{B}$ are models of T with domains $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$, respectively, and there is a function $f : M \to M$ such that for every n there is a tuple $\langle i_0, \ldots, i_{n-1}\rangle$ such that $i_j \leq f(j)$ for all j and such that*

$$\mathsf{tp}^{\mathcal{B}}(b_0, \ldots, b_{n-1}) = \mathsf{tp}^{\mathcal{A}}(a_{i_0}, \ldots, a_{i_{n-1}}),$$

then there is an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$.

Proof. Let $(M,)$ be a model of $\mathsf{WKL_0}$, and fix $T, \mathcal{A}, \mathcal{B}, f \in$ as in the hypothesis. We build a tree $\mathcal{T} \in$ such that any path through $\mathcal{T}$ can be used to define an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$ in a Δ^0_1 way. We then argue that $\mathcal{T}$ is infinite, and obtain the desired path using Weak König's Lemma. Let

$$\mathcal{U} = \{\sigma \in M^{<M} : \sigma(i) \leq f(i) \text{ for all } i < |\sigma|, \text{ and } \sigma(i) \neq \sigma(j) \text{ whenever } i \neq j\}.$$

Then $\mathcal{U}$ is a tree, and the infinite paths through $\mathcal{U}$ are exactly the injections $h : M \to M$ such that $h(n) \leq f(n)$ for all n. For each n, let $\{\phi^{(n)}_0, \phi^{(n)}_1, \ldots\}$ be an enumeration of all n-ary L-formulas. We define $\mathcal{T}$ to be the following subtree of $\mathcal{U}$:

$$\mathcal{T} = \{\sigma \in \mathcal{U} : (\forall i, n < |\sigma|)[\mathcal{B} \models \phi^{(n)}_i(b_0, \ldots, b_{n-1}) \text{ iff } \mathcal{A} \models \phi^{(n)}_i(a_{\sigma(0)}, \ldots, a_{\sigma_{n-1}})]\}.$$

If α is an infinite path of $\mathcal{T}$, then the function $g : \{b_0, \ldots\} \to \{a_0, \ldots\}$ given by $g(b_i) = a_{\alpha(i)}$ is an elementary embedding.

It remains to check that $\mathcal{T}$ is infinite. Fix any $n \in M$. By hypothesis, there is a tuple $(a_{i_0}, \ldots, a_{i_{n-1}})$ such that $i_j \leq f(j)$ for each j, and such that $\mathsf{tp}^{\mathcal{B}}(b_0, \ldots, b_{n-1}) = \mathsf{tp}^{\mathcal{A}}(a_{i_0}, \ldots, a_{i_{n-1}})$. Then the string σ of length n with $\sigma(j) = i_j$ is in $\mathcal{T}$. Hence $\mathcal{T}$ has at least n elements, as required. $\square$

Proof of Theorem 3.2.27. Fix a model $(M,)$ of $\mathsf{WKL_0}$, and fix $T, f \in$ such that T is a complete consistent theory, and $f : M \to M$ is a function such that $f(n)$ is greater than the number of n-types of T for all n. We must show that T has a universal model. To do this, we define a sequence $X = \langle p_0, \ldots \rangle$ of types of T such that every n-type is equal to p_i for some $i < 2^{f(0)} + 2^{f(1)} + \cdots + 2^{f(n)}$. We then let $\mathcal{A}$ be the model constructed as in the proof of Corollary 3.3.4 above, and use Lemma 3.3.5 to argue that $\mathcal{A}$ is universal.

For each $n \in M$, let $(\phi^{(n)}_t)_t$ be an enumeration of all n-ary L-formulas. We describe how to build a tuple $\langle q_0, \ldots, q_{2^{f(n)}-1} \rangle$ of n-types which includes every n-

type of T. Let $q_{k,0} = \emptyset$ for all k. If $\langle q_{0,s}, \ldots, q_{2^{f(n)}-1,s} \rangle$ is defined, let $q_{k,s+1} = q_{k,s} \cup \{\phi_s\}$ for exactly half of all k such that $T \not\vdash \bigwedge q_{k,s} \to \neg\phi_s$; let $q_{k,s+1} = q_{k,s} \cup \{\neg\phi(s)\}$ for all other k. Clearly each $q_k = \bigcup_s q_{k,s}$ is an n-type of T, and the tuple $\langle q_0, \ldots, q_{2^{f(n)}-1} \rangle$ exists by Δ^0_1 comprehension. To see that $\langle q_0, \ldots, q_{2^{f(n)}-1} \rangle$ contains all n-types, it is enough to notice that each n-type $p = \{\psi_0, \psi_1, \ldots\}$ contains at most $f(n)$ distinct ψ_m such that $T \not\vdash \bigwedge_{i<m} \psi_i \to \psi_m$.

Now iterate this method for all $n \in M$ to produce a sequence $X = \langle p_0, p_1, \ldots \rangle$ of types of T such that the first $2^{f(0)}$-many are a list of all 0-types,[2] the next $f(1)$-many are a list of all 1-types, and so on. Then X is an enumeration of all types of T; let $\mathcal{A}$ be the model produced in the proof of Corollary 3.3.4 using this X. Using the bound $2^{f(0)} + \cdots + 2^{f(n)}$ and the mapping from $\bar{p}, q, \bar{r}$ to $c_{\langle \bar{p}, q, \bar{r} \rangle}$ in the definition of $\Phi_{H,s}$, we can define a function $g : M \to M$ as in the hypothesis of Lemma 3.3.5. We conclude by that Lemma that $\mathcal{A}$ is a universal model of T. $\square$

Note that Lemma 3.3.5 can also be used to get a shorter, less explicit proof of Theorem 3.2.25. Moving on: this section's final result constructs not one, but a sequence of models. Its proof is based on a construction found in Chapter 2 and partially duplicates a theorem from there.

Proposition 3.3.6. $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ *proves the following. Fix a complete theory T which has infinitely many n-types for some n. If $(P, \leq)$ is a partial order with $P = \{p_0, p_1, \ldots\}$, then there is a sequence $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ of models of T such that $p_i \leq p_j$ if and only if $\mathcal{A}_i$ embeds elementarily into $\mathcal{A}_j$.*

Proof. Let $(M,)$ be a model of $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$. By Lemma 3.1.3, we may fix a set $Z \in$ whose Turing jump K^Z is not in . Fix a complete consistent theory $T \in$ a partial order $(P, \leq) \in$ with $P = \{p_0, p_1, \ldots\}$ with a number n as in the theorem statement.

[2] That is, if $i < 2^{f(0)}$, then $p_i = T$.

Consider the set $\mathcal{H}^\dagger = \{\langle\sigma_0, \ldots, \sigma_{k-1} \in \mathcal{H}^{<M} :$ each σ_i has length $|\sigma_i| = k\}$ with the ordering $\langle\sigma_0, \ldots, \sigma_{k-1}\rangle \prec \langle\tau_0, \ldots, \tau_{\ell-1}\rangle$ if $k \leq \ell$ and $\sigma_i \subseteq \tau_i$ for every $i < k$. This $\mathcal{H}^\dagger$ is an infinite tree, any path of which encodes a sequence $\langle\mathcal{B}_0, \ldots\rangle$ of models of T. What's more, $\mathcal{H}^\dagger$ can be encoded homeomorphically as a binary branching tree in a Δ^0_1 way. Similar to other proofs in this section, we define an infinite subtree of $\mathcal{H}^\dagger$ such that any $\langle\mathcal{B}_0, \ldots\rangle$ encoded by one of its paths satisfies the theorem, and then apply $\mathsf{WKL_0}$.

We have two sorts of requirement to meet. First, given i, j such that $p_i \leq p_j$, we must ensure that $\mathcal{B}_i$ embeds elementarily into $\mathcal{B}_j$. Second, given i, j such that $p_i \not\leq p_j$, we must ensure that $\mathcal{B}_i$ does not embed elementarily into $\mathcal{B}_j$. We address these two requirements separately, and then show how to combine the strategies to prove the theorem.

Making $\mathcal{B}_i$ embed into $\mathcal{B}_j$. Fix i and j. Let $(\psi_s)_{s\in M}$ be an enumeration of all L-formulas. Define a subtree $\mathcal{H}^\dagger_0$ of $\mathcal{H}^\dagger$ by:

$$\begin{aligned}\mathcal{H}^\dagger_0 = \{\langle\sigma_0, \ldots, \sigma_{k-1}\rangle \in \mathcal{H}^\dagger :\ &\text{if } T \cup D_{\sigma_i} \vdash \phi_s(a_0, \ldots, a_{m-1}) \\ &\text{then } T \cup D_{\sigma_j} \nvdash \neg\phi_s(a_{2\langle i,j,0\rangle}, \ldots, a_{2\langle i,j,m-1\rangle})\}.\end{aligned}$$

If $\langle\mathcal{B}_0, \ldots\rangle$ is encoded by a path in $\mathcal{H}^\dagger_0$, define a mapping from $\mathcal{B}_i$ to $\mathcal{B}_j$ by taking each $[c_k]_E$ in $\mathcal{B}_i$ to $[c_{2\langle i,j,k\rangle}]_E$ in $\mathcal{B}_j$. This is a Δ^0_1-definable elementary embedding.

Making $\mathcal{B}_i$ not embed into $\mathcal{B}_j$. Fix i and j. Our strategy is to ensure that the Turing jump K^Z is Δ^0_1-definable from any elementary embedding $\mathcal{B}_i \hookrightarrow \mathcal{B}_j$, and argue that $K^Z \notin$ implies no such embedding exists. We adapt the argument from Chapter 2. Let $(\phi_s)_s$ be an enumeration of all n-ary L-formulas. For each pair $\sigma, \tau \in \mathcal{H}$ and each natural number t, define an L^*-sentence $\theta_{\sigma,t}$ as follows.

- If there is an $s < t$ such that $T \vdash (\exists\bar{x})\phi_s(\bar{x})$, such that $T \cup D_\sigma \cup W_{|\sigma|} \vdash \neg\phi_s(\bar{d})$ for each n-tuple $\bar{d}$ from among constants $\{c_0, \ldots, c_{t-1}\}$, then let $\theta_{\sigma,t} = \phi_s$ for the least such s.

- Otherwise, let $\theta_{\sigma,t} = \mathsf{Tr}$ be the formal 'true' predicate.

Notice that if $\theta_{\sigma,t}$ is defined as in the first alternative and $\sigma \subseteq \tau$ then $\theta_{\tau,t} = \theta_{\sigma,t}$. Notice also that, if f is a path in $2^{<M}$ and t is a number, since T has infinitely many n-types, there is an initial segment $\sigma \subseteq f$ such that $\theta_{\sigma,t}$ is defined as in the first alternative. Furthermore, we can find this initial segment effectively. Define a subtree $\mathcal{H}_1^\dagger$ of $\mathcal{H}^\dagger$ by:

$$\mathcal{H}_1^\dagger = \{\langle \sigma_0, \ldots, \sigma_{k-1} \rangle \in \mathcal{H}^\dagger : \text{ if } \ell \in K_{t-1}^Z \text{ and } i, j, k < t$$

$$\text{then } T \cup D_{\sigma_i} \nvdash \neg\theta_{\sigma_j,t}(c_{2n\langle i,j,\ell\rangle}, \ldots, c_{2n(\langle i,j,\ell\rangle+1)-2})\}.$$

Let $\langle \mathcal{B}_0, \ldots \rangle$ be the sequence encoded by a path in $\mathcal{H}_1^\dagger$. Suppose for a contradiction that g is an elementary embedding from $\mathcal{B}_i$ to $\mathcal{B}_j$. Let ℓ and t be any pair such that $\ell \in K_{at\ t}^Z$. Then we have $\mathcal{B}_i \models \theta([c_{2n\langle i,j,\ell\rangle}]_E, \ldots, [c_{2n(\langle i,j,\ell\rangle+1)-2}]_E)$ and $\mathcal{B}_j \models \neg\theta([\bar{d}])$ for all n-tuples $\bar{d}$ taken from $\{c_0, \ldots, c_{t-1}\}$ where $\theta = \theta_{\sigma,t}$ for some σ. Hence g maps one of $[c_{2n\langle i,j,\ell\rangle}]_E, \ldots, [c_{2n(\langle i,j,\ell\rangle+1)-2}]_E$ to a $[c_s]_E$ with $s > t$. This allows us to define a function which dominates the modulus function for K^Z. It follows by Δ_1^0 comprehension that K^Z is an element of , a contradiction.

Combining the strategies. We combine the two strategies in a straightforward way. Define a subtree $\mathcal{H}^\ddagger$ of $\mathcal{H}^\dagger$ by:

$$\mathcal{H}^\ddagger = \bigcap_{p_i \leq p_j} \{\langle \sigma_0, \ldots, \sigma_{k-1} \rangle \in \mathcal{H}^\dagger : \text{ if } T \cup D_{\sigma_i} \vdash \phi_s(a_0, \ldots, a_{m-1})$$

$$\text{then } T \cup D_{\sigma_j} \nvdash \neg\phi_s(a_{2\langle i,j,0}, \ldots, a_{2\langle i,j,m-1\rangle})\}$$

$$\cap \bigcap_{p_i \not\leq p_j} \{\langle \sigma_0, \ldots, \sigma_{k-1} \rangle \in \mathcal{H}^\dagger : \text{ if } \ell \in K_{t-1}^Z \text{ and } i, j, k < t$$

$$\text{then } T \cup D_{\sigma_i} \nvdash \neg\theta_{\sigma_j,t}(c_{2n\langle i,j,\ell\rangle}, \ldots, c_{2n(\langle i,j,\ell\rangle+1)-2})\}.$$

It is not difficult to see that $\mathcal{H}^\ddagger$ is infinite and that, if $\langle \mathcal{B}_0, \ldots \rangle$ is the sequence of models encoded by a path, then by the arguments above $\mathcal{B}_i$ embeds elementarily into $\mathcal{B}_j$ if and only if $p_i \leq p_j$. We now obtain the desired $\langle \mathcal{B}_0, \ldots \rangle$ by applying WKL_0. $\square$

3.4 A Controlled failure of compactness

Recall from Lemma 3.1.5 that $\mathsf{WKL_0}$ is equivalent over $\mathsf{RCA_0}$ to the compactness theorem for first-order logic. The usual proof of the leftward direction of this equivalence begins by fixing a binary tree $\mathcal{T}$, and then building a complete theory T which satisfies the Compactness Theorem only if $\mathcal{T}$ has an infinite path. In this section, we give a construction that accomplishes roughly the same thing: it takes a tree and attempts to provide a counterexample to the compactness theorem. Yet this construction has certain advantages, namely, that it produces very intuitive models—in its most basic instance, it produces a theory where every singleton in every model is a definable set—and that it can be cleanly extended, as we do in §3.5.

The present section is laid out as follows. In §3.4.1, we detail a construction that transforms an infinite binary tree $\mathcal{T}$ into a complete theory T, and defines a certain sequence of unary predicates $\langle P_i \rangle_i$. Then, in §3.4.2, we show that, if $\mathcal{T}$ has no infinite path, the predicates P_i partition the universe of any model of T into infinitely many sets, each with the same cardinality. In particular, the set $\{\neg P_i(x) : i \in M\}$ of formulas is finitely satisfiable but not satisfiable.

To simplify the axioms and some steps of the verification, we build T indirectly as a reduct of another theory T^* on an expanded language. Our construction also has the odd feature that, for certain choices of binary tree $\mathcal{T}$, the theory T being built might be incomplete. It simplifies our analysis to assume from the start that $\mathcal{T}$ is an infinite tree with no infinite path, and, in particular, that $\mathcal{T}$ belongs to a model $(M,)$ of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$.

3.4.1 Construction

Fix a model $(M,)$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, an infinite binary tree $\mathcal{T} \in$ with no infinite paths, and a number $n \in M$. Let $\langle \tau_0, \tau_1, \ldots \rangle$ be a one-to-one listing of all terminal nodes of $\mathcal{T}$. Define a larger tree $\mathcal{T}_0$ by:

$$\mathcal{T}_0 = \mathcal{T} \cup \{\tau_i {}^\frown 0^s : \ i, s \in M\}.$$

Then $\mathcal{T}_0$ has no terminal nodes. Let $L = \langle R_\sigma : \sigma \in 2^{<M} \rangle$ be an infinite language of unary relations, and let $L^* = L \cup \langle c_{i,j} : i \in M, \ j < n \rangle$. Consider the following axiom schemes:

Ax I. $R_\emptyset(x)$.

Ax II. $R_\sigma(x) \to R_{\sigma {}^\frown 0}(x) \vee R_{\sigma {}^\frown 1}(x)$.

Ax III. $\neg(R_\sigma(x) \wedge R_{\sigma'}(x))$ whenever σ, σ' are incompatible strings.

Ax IV. $\neg R_\sigma(x)$ whenever σ is not an element of $\mathcal{T}_0$.

Ax V. $c_{i_0,j_0} \neq c_{i_1,j_1}$ whenever $i_0 \neq i_1$ or $j_0 \neq j_1$.

Ax VI. $R_\sigma(c_{i,j})$ whenever $\sigma \subseteq \tau_i$ and $j < n$.

Ax VII. $R_\sigma(x) \to \bigvee_{i \in F, j < n} x = c_{i,j}$ whenever F is a finite set containing all i such that $\sigma \subseteq \tau_i$.

Axioms I–IV say that, whenever $\mathcal{A}$ is a model of the axioms and a is an element, the set $\{\sigma : \mathcal{A} \models R_\sigma(a)\}$ forms a path through $\mathcal{T}$. By the definition of $\mathcal{T}_0$, this set is uniquely determined by the unique index i such that $\mathcal{A} \models R_{\tau_i}(a)$. Axiom V says simply that all the constants $c_{i,j}$ are distinct, and axioms VI–VII guarantee that those elements a for which $\mathcal{A} \models R_{\tau_i}(a)$ are exactly those given by constants $c_{i,0}, \ldots, c_{i,n-1}$. Despite their indirect definition, the axioms of Ax VII are a Δ^0_1 set; to see this, notice that every node $\sigma \in \mathcal{T}$ either has only finitely many extensions in $\mathcal{T}$, or has infinitely many terminal extensions.

Define a sequence of predicates P_i by:

$$P_i(x) \iff R_{\tau_i}(x).$$

We finish the construction by letting T^* be the deductive closure of the Ax I–VII, and letting T be the reduct of T^* to the language L. At this point, it is far from clear that T^* and T exist in the second-order part of $(M,)$; one of our main tasks in the verification below is to show that they do. This is accomplished below in Corollary 3.4.2.

3.4.2 Verification

We must verify that T is in the second-order part of $(M,)$, that it is a complete, consistent theory, and that the predicates P_i partition the universe of any model as outlined above. Since we have not yet proved that T or T^* exist in $(M,)$, we cannot bring to bear the usual model-theoretic tools, such as the Completeness Theorem. Instead, we must deal with formulas from first principles, by manipulating their syntax.

Lemma 3.4.1. *There is an algorithm which, given a conjunction of L^*-literals $\phi(\bar{x}, y)$, returns a quantifier-free L^*-formula $\psi(\bar{x})$ such that Axioms I–VII entail $\psi(\bar{x}) \leftrightarrow (\exists y)\phi(\bar{x}, y)$.*

Proof. Suppose that $\phi(\bar{x}, y)$ is a conjunction of L^*-literals, and let m be the length of $\bar{x} = \langle x_0, \ldots, x_{m-1}\rangle$. We may assume by Ax III, IV, and VI that no conjunct is of the form $R_\sigma(c_{i,j})$ or $\neg R_\sigma(c_{i,j})$; by Ax V, that none is of the form $c_{i_0,j_0} = c_{i_1,j_1}$ or $c_{i_0,j_0} \neq c_{i_1,j_1}$; by substituting variables, that none is of the form $t_0 = t_1$ for any terms z_0, z_1; and, by symmetry of $=$, that none is of the form $c_{i,j} \neq z$ for any variable z. (The remaining conjuncts are of the form $R_\sigma(z)$, $\neg R_\sigma(z)$, and $z \neq c_{i,j}$, where z is a variable and $c_{i,j}$ is a constant.)

Let $\phi_0(y)$ be the formula obtained by replacing with Tr every conjunct mentioning any x_k, $k < m$. Then ϕ_0 is a conjunction of literals of the forms Tr, $R_\sigma(y)$, $\neg R_\sigma(y)$, and $y \neq c_{i,j}$. If $\langle i, j\rangle$ is a pair such that $\tau_i \supseteq \sigma$ for each $R_\sigma(y)$ in ϕ_0, such that $\tau_i \not\supseteq \sigma$ for each $\neg R_\sigma(y)$ in ϕ_0, and such that $y \neq c_{i,j}$ is not in ϕ_0, then Axioms I–VII imply $\phi_0(c_{i,j})$; otherwise, they imply $\neg\phi_0(c_{i,j})$. We can check effectively—using the fact that $\mathcal{T}$ has no infinite path—whether there exist more that m distinct such pairs.

Case 1: There are distinct such pairs $\langle i_0, j_0\rangle, \ldots, \langle i_m, j_m\rangle$. Let $\psi(\bar{x})$ be the formula:

$$\psi(\bar{x}) \Leftrightarrow \phi(\bar{x}, c_{i_0,j_0}) \vee \cdots \vee \phi(\bar{x}, c_{i_m,j_m}).$$

The implication $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ is a tautology. We now show that Ax I–VII prove the converse statement $(\exists y)\phi(\bar{x}, y) \to \psi(\bar{x})$. Let $\phi_1(\bar{x})$ be the formula obtained from ϕ by replacing with Tr each conjunct mentioning y. Then $\phi(\bar{x}, y)$ is equivalent to the formula

$$\phi_1(\bar{x}) \wedge \phi_0(y) \wedge \bigwedge_{\ell \in E} x_\ell \neq y$$

for some set $E \subseteq \{0, \ldots, m-1\}$. Of course, $(\exists y)\phi(\bar{x}, y) \to \phi_1(\bar{x})$ is a tautology, and $\phi_0(c_{i_k,j_k})$ is true for each k by choice of i_k, j_k. As well, by the Pigeonhole Principle, Axiom V is enough to prove

$$\bigvee_{\ell \leq m} \bigwedge_{k<m} x_k \neq c_{i_\ell,j_\ell}.$$

Hence Ax I–VII can prove

$$(\exists y)\phi(\bar{x}, y) \to \bigvee_{\ell \leq m} \left(\phi_1(\bar{x}) \wedge \phi_0(c_{i_k,j_k}) \wedge \bigwedge_{k \notin E} x_\ell \neq c_{i_k,j_k} \right),$$

which is equivalent to the desired statement.

Case 2: There are no more than m distinct such pairs. Let $\langle i_0, \ldots, i_{\ell-1}\rangle$ be a list of all i such that $\tau_i \supseteq \sigma$ whenever $R_\sigma(y)$ is in ϕ_0 and $\tau_i \not\supseteq \sigma$ whenever $\neg R_\sigma(y)$

is in ϕ_0. Axioms II and IV prove the statement

$$\phi_0(y) \to \bigvee_{i<\ell} P_i(y).$$

Together with Axiom VII, this gives

$$\phi_0(y) \to \bigvee_{k<\ell} \bigvee_{j<n} y = c_{i_k,j}. \tag{3.1}$$

Now let $\psi(\bar{x})$ be the formula:

$$\psi(\bar{x}) \Leftrightarrow \bigvee_{k<\ell} \bigvee_{j<n} \phi(\bar{x}, c_{i_k,j}).$$

The implication $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ is a tautology. The converse implication $(\exists y)\phi(\bar{x}, y) \to \psi(\bar{x})$ follows from the displayed formula (3.1). $\square$

Corollary 3.4.2. *The deductive closure T^* of Ax I–VII exists in $(M,)$, and admits effective quantifier elimination. The reduct T of T^* to the language L exists in $(M,)$.*

Proof. If ϕ is an L^*-sentence, we can apply the effective procedure from Lemma 3.4.1 iteratively to obtain a quantifier-free L^*-sentence ψ such that Ax I–VII entail $\phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. (See Lemma 2.1.7 or [30, Proof of Thm 2.3].) Since Ax I–VII decide every quantifier-free L^*-sentence—it is clear which constants satisfy which relations—it follows that Ax I–VII decide every L^*-sentence.

Therefore the theory T^* exists by Δ^0_1 comprehension, as does its reduct T. $\square$

Lemma 3.4.3. *T^* is consistent.*

Proof. We begin defining a model $\mathcal{A}^*$ with universe $A = \langle a_{i,j} : i \in M \text{ and } j < n\rangle$ by specifying its *atomic* diagram, beginning with:

$\mathcal{A}^* \models a_{i,j} = c_{i,j}$, and $\mathcal{A}^* \models a_{i_0,j_0} \neq a_{i_1,j_1}$ whenever $\langle i_0, j_0\rangle \neq \langle i_1, j_1\rangle$, and $\mathcal{A}^* \models R_\sigma(a_{i,j})$ if and only if $\sigma \subseteq \tau_i$ or $\sigma = \tau_i{}^\frown 0^s$ for some s.

This atomic diagram satisfies each of Ax I–VII. Use the effective procedure for quantifier elimination given by Lemma 3.4.2 to assign a truth value to every $L^* \cup \{a_0, \ldots\}$-sentence for $\mathcal{A}^*$. To see that we end up with an elementary diagram—that is, a set free of inconsistencies and closed under entailment—notice first that, by the derivation of our effective procedure, every ϕ with quantifier depth 1 is assigned a truth value that is *semantically* correct from the atomic diagram. It follows by Δ^0_1 induction that every sentence's truth value is semantically true, giving the desired consistency and closure properties. (A formal development of the semantic side of first-order logic can be found in [63, section II.8].) □

Corollary 3.4.4. *T^* and T are complete, consistent theories.*

And at last we can check some less basic properties of T. Recall that n is a natural number fixed in §3.4.1 and used in defining the axioms of T^*.

Lemma 3.4.5. (i) *If $\mathcal{A}^*$ is a model of T^*, the sets $P_i^{\mathcal{A}^*} = \{a : \mathcal{A}^* \models P_i(a)\}$ partition its domain. Furthermore, each of these sets has size n.*

(ii) *If $\mathcal{A}^*$ is a model of T^*, each element is equal to some constant $c_{i,j}^{\mathcal{A}^*}$.*

(iii) *If $\mathcal{A}$ is a model of T, then the sets $P_i^{\mathcal{A}} = \{a : \mathcal{A} \models P_i(a)\}$ partition its domain into sets of size n.*

Proof. (i) Because $\mathcal{T}$ has no infinite path, Ax I–IV ensure that for each element a there is a unique terminal node τ_i of $\mathcal{T}$ such that $\mathcal{A}^* \models P_i(a)$. Hence the sets $P_i^{\mathcal{A}^*}$ partition the domain. If a is an element and τ_i is the corresponding terminal node, then by Axiom VII we know that $\mathcal{A}^* \models a = c_{i,j}$ for some $j < n$. It follows that $P_i^{\mathcal{A}^*}$ is equal to the set $\{c_{i,j}^{\mathcal{A}^*} : j < n\}$. By Axiom V, these $c_{i,j}^{\mathcal{A}^*}$ are all distinct, so $P_i^{\mathcal{A}^*}$ has size n.

(ii) Already proved as part of 3.4.5.

(iii) Each of Axioms I–IV uses only symbols from L, and so is contained in T. As in 3.4.5, this means the sets $P_i^{\mathcal{A}}$ partition the domain of $\mathcal{A}$. What's more, by 3.4.5 we know that the formula $(\exists^{=n} x)P_i(x)$ is contained in T^* and uses only symbols from L, and so is contained in T as well. It follows that each $P_i^{\mathcal{A}}$ has size n.

□

Lemma 3.4.6. *Suppose $\mathcal{A}$ is a model of T with domain A.*

(i) *There is a model $\mathcal{A}^*$ of T^* extending $\mathcal{A}$.*

(ii) *Any permutation of A taking each $P_i^{\mathcal{A}}$ back to $P_i^{\mathcal{A}}$ is an automorphism of $\mathcal{A}$.*

Proof. (i) Given $i \in M$, we may effectively find all n distinct elements a such that $\mathcal{A} \models P_i(a)$. Define $\mathcal{A}^*$ by letting $c_{i,0}^{\mathcal{A}^*}, \ldots, c_{i,n-1}^{\mathcal{A}^*}$ be a listing of these elements for each i. Extend to an elementary diagram as in the proof of Lemma 3.4.3.

(ii) Suppose that f is a permutation of the domain of $\mathcal{A}$ mapping each $P_i^{\mathcal{A}}$ back to $P_i^{\mathcal{A}}$. Let $\mathcal{A}_0^*$ be an extension of $\mathcal{A}$ as in part (i) above, and let $\mathcal{A}_1^*$ be another extension given by $c_{i,k}^{\mathcal{A}_1^*} = f(c_{i,k}^{\mathcal{A}_0^*})$ for each i, k. Then f is an isomorphism from $\mathcal{A}_0^*$ to $\mathcal{A}_1^*$.

□

3.4.3 Applications

The main application of this construction comes when we extend it in §3.5. For now, we now give a separate, immediate model-theoretic consequence.

Proposition 3.4.7. $\mathsf{RCA}_0 \vdash$ (*Every complete consistent theory has a model with a sequence of order indiscernibles*) $\to \mathsf{WKL}_0$.

Proof. We prove the contrapositive statement. Suppose $(M,)$ is a model of $\mathsf{RCA_0}+\neg\mathsf{WKL_0}$, let T be the theory constructed in §3.4.1 with $n = 1$, and let $\mathcal{A}$ be any model of T. Suppose for a contradiction that there is a sequence of order indiscernibles with distinct elements a and b. Then by Lemma 3.4.53.4.5, there is a j such that $\mathcal{A} \models P_j(a)$ and $\mathcal{A} \models \neg P_j(b)$, a contradiction. □

We also note in passing that, with a few minor changes to the axioms and verification, the construction in §3.4.1 gives a theory whose every model is partitioned into countably many *infinite* sets, or sets of different sizes.

Corollary 3.4.8 ($\mathsf{RCA_0}+\neg\mathsf{WKL_0}$)**.** *Let f be a total function $f : M \to \{1, 2, \ldots\} \cup \{\aleph_0\}$, where $\aleph_0$ is a formal symbol denoting a countable infinity. There is a complete consistent theory T with a sequence of unary formulas $P_0(x), \ldots$ with the following properties: If $\mathcal{A}$ is a model of T with universe A, then the sets $P_m^{\mathcal{A}}$ form a partition of A, with $|P_m^{\mathcal{A}}| = f(m)$ for all m, and any permutation of A fixing each $P_m^{\mathcal{A}}$ is an automorphism of $\mathcal{A}$.*

3.5 1-Homogeneity vs strong 1-homogeneity

In this section, we produce an example of a theory T with only principal types, but with no strongly 1-homogeneous model. This theory is built by extending the construction in §3.4 above. As such, we again work within a model $(M,)$ of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$, and construct T indirectly as a reduct of a larger theory T^*.

We begin with an outline of the construction and its verification. Fix a model $(M,)$ of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$. Recall from Definition 3.1.6 the notion of an *inseparable Σ^0_1 pair*. Using $\neg\mathsf{WKL_0}$ and Lemma 3.1.5, fix an inseparable Σ^0_1 pair $\langle U_s, V_s\rangle_s \in$. Let $L = \langle Q_s, B_s, R_\sigma\rangle_{s,\sigma}$ be the language where each Q_s and each R_σ is a unary relation symbol, and each B_s is a binary relation symbol. We design an L^*-theory

T^* so that, if $\mathcal{A}^*$ is a model of T^*, $\mathcal{A}$ is the reduct of $\mathcal{A}^*$ to L, and A is the domain of $\mathcal{A}^*$, then the following hold.

(B1) T^* includes all the axioms listed in the construction of §3.4.1 with $n = 2$.

(B2) There is a sequence of L-formulas $P_0(x), P_1(x), \ldots$ such that the sets $P_i^{\mathcal{A}}$ form a partition of A. Furthermore, each set $P_i^{\mathcal{A}}$ consists exactly of the elements $c_{k,0}^{\mathcal{A}^*}$ and $c_{k,1}^{\mathcal{A}^*}$.

(B3) The elements $c_{0,0}^{\mathcal{A}^*}$ and $c_{0,1}^{\mathcal{A}^*}$ satisfy the same L-formulas. (In other words, $c_{0,0}^{\mathcal{A}^*}$ and $c_{0,1}^{\mathcal{A}^*}$ realize the same 1-type in $\mathcal{A}$.)

(B4) Any automorphism of $\mathcal{A}$ which maps $c_{0,0}^{\mathcal{A}^*}$ to $c_{0,1}^{\mathcal{A}^*}$ computes a separating set for $\langle U_s, V_s \rangle_s$. (Hence no such automorphism exists in .)

We now give a hint as to what the structures $\mathcal{A}^*$ and $\mathcal{A}$ look like. As mentioned in property (B2), there is a sequence $P_0, \ldots$ of unary predicates which partition A into sets of size 2, with each $P_s^{\mathcal{A}}$ consisting of the elements $c_{k,0}^{\mathcal{A}^*}$ and $c_{k,1}^{\mathcal{A}^*}$. The unary predicate Q_s holds of an element $a \in A$ if and only if $a = c_{k+1,0}^{\mathcal{A}^*}$ where $k \in U_{at\ s}$. The binary predicate B_s holds of a pair $a, b \in A$ if and only if both $a = c_{0,j}^{\mathcal{A}^*}$ and $b = c_{k+1,j}^{\mathcal{A}^*}$ are true when $k \in V_{at\ s}$ and some $j \in \{0, 1\}$.

3.5.1 Construction

Fix a model $(M,)$ of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$. Fix an infinite tree $\mathcal{T} \in$ with no infinite path. Let $L = \{Q_s, R_\sigma, B_s : s \in M, \sigma \in 2^{<M}\}$ be a relational language where each Q_s, R_σ is unary and each B_s is binary. Let $L^* = L \cup \{c_{i,j} : i \in M, j \in \{0, 1\}\}$, where each $c_{i,j}$ is a constant symbol. Consider the following axiom schemes:

$$\left.\begin{array}{r} \text{Ax I.} \\ \vdots \\ \text{Ax VII.} \end{array}\right\} \text{As in §3.4.1, with } n = 2.$$

Ax VIII. $Q_s(c_{k+1,0})$ if k enters U at stage s.

Ax IX. $\neg Q_s(c_{k,j})$ for all other choices of j, k, s.

Ax X. $B_s(c_{0,j}, c_{k+1,j})$ for each j, k, s such that k enters V at stage s.

Ax XI. $\neg B_s(c_{k_0,j_0}, c_{k_1,j_1})$ for all other choices of j_0, j_1, k_0, k_1, s.

We now give the intuition behind the axioms, in terms of the properties (B1–B2) listed near the beginning of this section. The first seven are exactly the axioms used in the construction of §3.4.1 above when $n = 2$, so (B1) is true. It follows by Lemma 3.4.53.4.5 that (B2) holds as well. Axioms VIII–XI give property (B4)—see Lemma 3.5.5 below. The remaining property (B3) holds because, roughly speaking, the axioms treat $c_{0,0}$ an $c_{0,1}$ symmetrically—see Lemma 3.5.2(iv) below for the details.

Use $\mathcal{T}$ and the relations R_σ to define a sequence of unary predicates P_i as in §3.4.1. Finish the construction by letting T^* be the deductive closure of the Ax I–XI and T the reduct of T^* to L; as in §3.4.1, it is not yet clear that T^* and T should exist in $(M,)$. We deal with this early in the verification as part of Lemma 3.5.1.

3.5.2 Verification

We begin by listing some basic properties of T and T^* such as existence and completeness. The proofs are analogous to those in §3.4.

Lemma 3.5.1. (i) *There is an algorithm which, given a conjunction of L-literals $\phi(\bar{x}, y)$, returns a quantifier-free L^*-formula $\psi(\bar{x})$ such that Ax I–XI prove $\psi(\bar{x}) \leftrightarrow (\exists y)\phi(\bar{x}, y)$.*

(ii) *T^* exists in $(M,)$ and has effective quantifier elimination.*

(iii) *T exists.*

(iv) *T^* is consistent. T is consistent. T is complete.*

Proof. (i) Similar to Lemma 3.4.1.

(ii) Follows from (i), similar to Corollary 3.4.2.

(iii) Follows from part (ii) and Δ^0_1 comprehension.

(iv) Similar to Lemma 3.4.3: Find the unique structure $\mathcal{A}^* \models T^*$ with universe $\{a_{i,j} : i \in M,\ j \in \{0,1\}\}$ such that $\mathcal{A} \models a_{i,j} = c_{i,j}$ for each i, j.

$\square$

Next, some less basic properties.

Lemma 3.5.2. (i) *If $\mathcal{A}^*$ is a model of T^*, then the predicates P_i partition its domain into sets $P_i^{\mathcal{A}^*}$ of size 2. Furthermore, $P_i^{\mathcal{A}^*}$ is equal to $\{c_{i,0}^{\mathcal{A}^*}, c_{i,1}^{\mathcal{A}^*}\}$ for all i. Hence property (B2) holds.*

(ii) *If $\mathcal{A}$ is a model of T, then the sets $P_i^{\mathcal{A}}$ partition its domain into sets of size 2.*

(iii) *Every 1-type of T is principal.*

(iv) *Every type of T is principal.*

(v) *Every model of T is 1-homogeneous.*

Proof. (i) Similar to Lemma 3.4.5(i).

(ii) Similar to Lemma 3.4.5(iii).

(iii) Fix a 1-type $p(x)$ of T. By Lemma 3.5.2(ii), Lemma 3.5.1(iii), and the Completeness Theorem, there is a j such that p contains $P_j(x)$ and $T \vdash (\exists^{=2} y) P_j(y)$. So either $P_j(x)$ generates $p(x)$, or there is a $\phi(x)$ such that $\phi(x) \to P_j(x)$ is a tautology, p contains $(\exists^{=1} y)\phi(y)$, and $\phi(x)$ generates $p(x)$.

(iv) Fix an n-type $p(\bar{x}) = p(x_0, \dots, x_{n-1})$ of T. Identifying variables if necessary, we may assume that $x_i \neq x_j$ is in $p(\bar{x})$ for every pair $i \neq j$. We know from Lemma 3.5.1 that for each $k < n$ there is an i_k such that $P_{i_k}(x_k)$ is in $p(\bar{x})$, and $T \vdash (\exists^{\leq 2} y) P_{i_k}(y)$. Let $\psi(\bar{x})$ denote the conjunction $P_{i_0}(x_0) \wedge \dots \wedge P_{i_{n-1}}(x_{n-1})$. Then $\psi(\bar{x})$ is in $p(\bar{x})$, and $T \vdash (\exists^{\leq 2^n} \bar{x}) \psi(\bar{x})$.

Using $\mathsf{I\Sigma}^0_1$, let $k \leq n$ be greatest such that there is a formula $\phi(\bar{x})$ with $T \vdash \phi(\bar{x}) \to \psi(\bar{x})$ and $T \vdash (\exists^{\leq k} \bar{x}) \phi(\bar{x})$. We claim $\phi(\bar{x})$ generates p. For a contradiction, suppose that it does not, i.e., suppose there is $\theta \in p$ such that $T \vdash (\exists \bar{x})[\phi(\bar{x}) \wedge \neg\theta(\bar{x})]$. Then $T \vdash (\exists^{\leq k-1} \bar{x})[\phi(\bar{x}) \wedge \theta(\bar{x})]$, and $\phi(\bar{x}) \wedge \theta(\bar{x})$ is in p, contradicting the minimality of k.

(v) Immediate from (iv).

□

Now we wish to show that no model $\mathcal{A}$ of T is strongly 1-homogeneous. We begin by showing that T admits a restricted form of quantifier elimination, classically equivalent to model completeness.

Lemma 3.5.3. *Every L-formula is equivalent over T to an $\exists$ L-formula.*

Proof. Fix an L-formula $\phi(\bar{x})$. Using the quantifier elimination from Lemma 3.5.1(ii), fix a quantifier-free L^*-formula $\psi(\bar{x})$ such that $T^* \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Our goal is to find an $\exists$ formula $\sigma(\bar{x})$ in the language L such that $T^* \vdash \phi(\bar{x}) \leftrightarrow \sigma(\bar{x})$. Let $\psi[\bar{y}/\bar{c}]$ denote the L-formula obtained by replacing each occurrence of a constant $c_{m,j}$ in ψ, with a new variable $y_{m,j}$. ($\psi[\bar{y}/\bar{c}]$ has free variables $(\bar{x}, \bar{y})$.)

Now let $p(\bar{y})$ be any type containing $y_{m,j} = c_{m,j}$ for each m, j. Using Lemma 3.5.2(iv), choose an L-formula $\theta(\bar{y})$ which generates p. Define an L-formula $\sigma(\bar{x})$ by

$$\sigma(\bar{x}) \Leftrightarrow (\exists \bar{y}) \theta(\bar{y}) \wedge \psi[\bar{y}/\bar{c}].$$

We claim $T \vdash \phi(\bar{x}) \leftrightarrow \sigma(\bar{x})$. The forward direction $\phi(\bar{x}) \rightarrow \sigma(\bar{x})$ is clearly in T^*, so it is in the reduct T as well. To see that the reverse direction $\sigma(\bar{x}) \rightarrow \phi(\bar{x})$ is in T, simply note that the sentence $(\forall \bar{x})(\phi(\bar{x}) \leftrightarrow \psi[\bar{y}/\bar{c}]$ is in p. □

The following two lemmas show that T has no strongly 1-homogeneous model.

Lemma 3.5.4. *The predicate $P_0(x)$ generates a principal 1-type of T. Hence property (B3) holds.*

Proof. It is clear from the axioms that, for every unary $\exists$ L-formula $\phi(x)$, either $T \vdash P_0(x) \rightarrow \phi(x)$ or $T \vdash P_0(x) \rightarrow \neg\phi(x)$. It follows by Lemma 3.5.3 that $P_0(x)$ generates a 1-type. □

Lemma 3.5.5. *Fix any model $\mathcal{A}$ of T.*

(i) *If f is an automorphism of $\mathcal{A}$ which swaps the two elements of $P_0^{\mathcal{A}}$, then there is a separating set for $\langle U_s, V_s \rangle_s$ which is Δ^0_1 definable from f.*

(ii) *There is no automorphism of $\mathcal{A}$ which swaps the two elements of $P_0^{\mathcal{A}}$.*

Proof. (i) Enumerate the elements of $\mathcal{A}$ as $a_{0,0}, a_{0,1}, \ldots, a_{i,0}, a_{i,1}, \ldots$, with $P_i^{\mathcal{A}} = \{a_{i,0}, a_{i_1}\}$ for every i. Suppose f is an automorphism of $\mathcal{A}$ such that $f(a_{0,0}) = a_{0,1}$. Define a set C to be all $k \in M$ such that f swaps the elements of $P_{k+1}^{\mathcal{A}}$, that is,

$$C = \{k : f(a_{k+1,0}) = f(a_{k+1,1})\}.$$

For every k, s such that $k \in U_s$, we must have $k \notin C$ by Axiom VIII; and for every k, s such that $k \in V_s$, we must have $k \in C$ by Axiom X. Hence C is a separating set for $\langle U_s, V_s \rangle_s$.

(ii) Follows from (i) and our choice of $\langle U_s, V_s \rangle_s$ as an inseparable Σ^0_1 pair.

□

3.5.3 Application

The following completes the proof of Theorem 3.2.4:

Proposition 3.5.6. $\mathsf{RCA}_0 \vdash$ (*Every complete theory with all types principal has a strongly 1-homogeneous model*) $\to \mathsf{WKL}_0$.

Proof. We show the contrapositive. Suppose that $(M,)$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, and let T be as in §3.5.1. Then T is complete, by Lemma 3.5.1; has all types principal, by Lemma 3.5.2(iv); and is not strongly 1-homogeneous, by Lemmas 3.5.4 and 3.5.5. □

3.6 A theory with the finite free amalgamation property, but without the 1-point full amalgamation property

In this section, we construct, in a model $(M,)$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, a theory with countably many types, and an enumeration of all types with the finite free amalgamation property but without the 1-point full amalgamation property. Our method is a very slight twist on Millar's [47] construction in effective mathematics of a decidable theory with exactly two decidable models up to recursive isomorphism, which was formalized in reverse mathematics in Chapter 2. The changes from Chapter 2 are minor: we add two new relations, a unary C and a binary E; we include axioms stating that E is an equivalence relation partitioning the domain into infinitely many infinite classes; and we require that E hold of a pair (x, y) whenever any other binary relation R_k holds of (x, y). Because the differences are so slight, we leave much of the verification as a sketch.

3.6.1 Construction

Work in a model $(M,)$ of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$. Let $\langle U_s, V_s\rangle_s$ be an inseparable Σ^0_1 pair. Let $L = \{P_s, R_k, C, E : s, k \in M\}$ be the relational language where P_s, C are unary and E, R_k are binary for all k, s. Consider the following axiom schemes.

Ax I. E is an equivalence relation.

Ax II. $P_s(x) \to P_t(x)$, whenever $t \leq s$

Ax III. $R_k(x, y) \to (E(x, y) \wedge P_k(x) \wedge P_k(y) \wedge x_i \neq x_j)$.

Ax IV. $(E(x, y) \wedge P_s(x) \wedge P_s(y) \wedge x \neq y) \to R_k(x, y)$, whenever $k \in U_s$.

Ax V. $(P_s(x) \wedge P_s(y) \wedge x \neq y) \to \neg R_k(x, y)$, whenever $k \in V_s$.

Ax VI. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L_0-literals, where $L_0 = \{E, P_i, R_i, C : i < k\}$ for some k;
- $\phi(\bar{x}, y)$ is consistent with Ax I–V;
- $\phi(\bar{x}, y) \to \psi(\bar{x})$ is a tautology;
- For each atomic L_0-formula θ with variables taken from $\bar{x}$, either θ or $\neg\theta$ is a conjunct in ψ; similarly, each atomic L_0-formula with variables from $\bar{x}, y$ or its negation is a conjunct in ϕ.

Let T be the deductive closure of Ax I–VI. This completes the construction. Note that we have not shown T is an element of ; this is accomplished as part of Lemma 3.6.1 below.

3.6.2 Verification

The following properties can each be verified in RCA_0 by altering the appropriate lemma from Chapter 2:

Lemma 3.6.1. (i) *T is an element of . T is complete.*

(ii) *T is consistent.*

(iii) *T has exactly two nonprincipal 1-types $q_0(x)$ and $q_1(x)$.*

(iv) *T has countably many types.*

(v) *If $\mathcal{A}$ is a model of T with elements a_0 and a_1 realizing $p_0(x)$, $p_1(x)$, respectively, then $\mathcal{A} \models \neg E(a_0, a_1)$.*

Let X be the enumeration of all types of T produced in Lemma 3.6.1(iv).

Lemma 3.6.2. *X has the finite free amalgamation property.*

Proof. Suppose that $\langle p_{i_0}, \ldots, p_{i_{n-1}} \rangle$ is a tuple of types in X, no two of which share a variable. Then it is easy to produce a type q extending

$$p_{i_0} \cup \cdots \cup p_{i_{n-1}} \cup \{\neg E(x, y) : x \text{ is a variable of } p_{i_j},\ y \text{ is a variable of } p_{i_k},\ j \neq k\}.$$

□

Lemma 3.6.3. *X does not have the 1-point full amalgamation property*

Proof. Let $q_0(y), q_1(z)$ be the distinct nonprincipal 1-types from Lemma 3.6.1(iii). Let $p(x)$ be the principal 1-type generated by $\neg P_0(x)$. Then there are 2-types $r_0(y) \supseteq p(x) \cup q_0(y) \cup \{E(x, y)\}$ and $r_1(z) \supseteq p(x) \cup q_1(z) \cup \{E(x, z)\}$. Suppose for a contradiction that X has the 1-point full amalgamation property. Then there is a 3-type $s(x, y, z)$ extending both r_1 and r_2. Let $\mathcal{A}$ be a model realizing s, say with $s(a, b, c)$ holding. Then $q_0(b)$ holds, $q_1(c)$ holds, and $\mathcal{A} \models E(b, c)$. But this is impossible by Lemma 3.6.1(v). □

3.6.3 Applications

Proposition 3.6.4. $\mathsf{RCA}_0 \vdash$ (*If X is an enumeration of all types of a complete consistent theory T and X has the finite free amalgamation property, then X has the 1-point full amalgamation property*) $\to \mathsf{WKL}_0$.

Proof. We prove the contrapositive. Suppose that $\neg\mathsf{WKL}_0$ holds, and let T be the theory constructed in §3.6.1, and let X be the sequence of all types described in the proof of Lemma 3.6.1(iv). We know from §3.6.2 that T is a complete consistent theory, and that X has the finite free amalgamation property but not the 1-point full amalgamation property. □

Proposition 3.6.5. $\mathsf{RCA}_0 \vdash$ (*Every complete consistent theory with countably many types has a saturated model*) $\to \mathsf{WKL}_0$.

Proof. Follows from Proposition 3.6.4 and Theorem 3.2.7. □

3.7 The case with neither WKL_0 nor Σ^0_2 induction

Our goal in this section is to complete the proofs of Theorem 3.2.14 and Theorem 3.2.23. We do this by constructing, within a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$, a pair of complete consistent theories. The first (§3.7.3) is a theory with an enumeration of all types which has the 1-point full, but not the pairwise free, amalgamation property. This is enough to complete the proof of Theorem 3.2.23(i). The second (§3.7.4) is a theory with an enumeration of all types which has the pairwise full, but not the finite free, amalgamation property. This is enough to prove Theorem 3.2.23(ii) and, after we introduce Lemma 3.7.16 below, to complete the proof of Theorem 3.2.14.

The basic idea is as follows. Working within a model $(M,)$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$, let $\langle U_s, V_s\rangle_s$ be an inseparable Σ^0_1 pair, as given by Lemma 3.1.5, and let

$\langle D_1 \subseteq D_2 \subseteq \cdots \rangle$ be a counterexample to $\mathsf{I}\Sigma^0_2$ as given by Lemma 3.1.9(iv). We use these $\langle U_s, V_s \rangle$ and D_i to define a theory T, along with a finite sequence $\langle p_0(x), \ldots, p_{n-1}(x) \rangle$ of nonprincipal 1-types. These p_i witness the failure of the appropriate amalgamation property in both of our theories; which amalgamation properties hold and which fail depends on the specifics of the sequence $\langle D_1 \subseteq \cdots \rangle$.

The construction is based loosely on the same paper of Millar's [47] as that in §3.6 above.

3.7.1 Construction

We work in a model $(M,)$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$. By Lemma 3.1.9, we may fix a coded sequence $D_1 \subseteq D_2 \subseteq \cdots$ of finite sets such that D_1 is finite, D_n finite implies D_{n+1} finite, and such that D_N is infinite for some N.

Let L be the relational language $L = (P_s, R^k_s, C_k)_{s \in M, k<N}$, where each P_s and C_k is unary, and each R^k_s is k-ary. Consider the following axiom schemes.

Ax I. $P_{s+1}(x) \to P_s(x)$.

Ax II. $R^k_s(x_0, \ldots, x_{k-1}) \to x_i \neq x_j$, whenever $i < j < k$.

Ax III. $R^k_s(x_0, \ldots, x_{k-1}) \to P_s(x_i)$, whenever $i < k$.

Ax IV. $\bigwedge_{i<k} P_s(x_i) \to R^k_\ell(x_0, \ldots, x_{k-1})$, whenever $s \in D_k$, ℓ is the m-th least element of D_k, and $m \in U_s$.

Ax V. $\bigwedge_{i<k} P_s(x_i) \to \neg R^k_\ell(x_0, \ldots, x_{k-1})$, whenever $s \in D_k$, ℓ is the m-th least element of D_k, and $m \in V_s$.

Ax VI. $\neg R^k_\ell$, whenever $\ell \notin D_k$.

Ax VII. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L_0-literals, where $L_0 = \{P_i, R_i, C_k : i < \ell\}$ for some ℓ;
- $\phi(\bar{x}, y)$ is consistent with Ax I–VI;
- $\phi(\bar{x}, y) \to \psi(\bar{x})$ is a tautology;
- For each atomic L_0-formula θ with variables taken from $\bar{x}$, either θ or $\neg\theta$ is a conjunct in ψ; similarly, each atomic L_0-formula with variables from $\bar{x}, y$ or its negation is a conjunct in ϕ.

Let T^{**} be the collection of all L-sentences in Ax I–VI, let T^* be the collection of all sentences in Ax I–VII, and let T be the deductive closure of T^*. This completes the construction. Notice that, although T^{**} is Δ^0_1 definable and therefore is an element of , we have not yet shown that either T^* or T is in ; this is accomplished as part of Lemma 3.7.2 below.

We now explain the intuition behind these axioms. Axioms I–III are analogous to the first three axioms of §3.6. Axioms IV and V are similar to the fourth and fifth axioms of §3.6 and push the relations P_s and R^k_ℓ towards encoding a separating set for $\langle U_s, V_s \rangle_s$, but they apply only to numbers ℓ, s which are in the appropriate D_k. Axiom VI keeps the remaining R^k_ℓ from taking on too many possible values (which is necessary if we expect T to have only countably many types). Lastly, Axiom VII gives quantifier elimination (part of Lemma 3.7.2 below). Notice that the relations C_k appear only in instances of Axiom VII.

3.7.2 Verification

Our first task is to show that T is an element of and is a complete, consistent theory. We begin with a simple, but technical, lemma.

Lemma 3.7.1. *Let L_0 be a relational language and $\Phi = \{(\forall \bar{x})\theta_0, (\forall \bar{y})\theta_1, \ldots\}$ a set of L_0-sentences, where each θ_n is quantifier-free and of the form $\psi_{0,n} \vee \psi_{1,n}$, where neither $\psi_{0,n}$ nor $\neg\psi_{0,n}$ is a tautology, and where no relation in $\psi_{0,n}$ appears in $\psi_{1,n}$ or in any θ_k, $k < n$. Then Φ is satisfiable, and there is a procedure that decides, given a quantifier-free L_0-formula ϕ, whether $\Phi \cup \{(\exists x)\phi\}$ is satisfiable.*

Proof. See Lemma 2.6.1. □

This allows us to verify some basic facts about T:

Lemma 3.7.2. (i) *The sentences in T^{**} can be rewritten so as to meet the conditions on Φ in Lemma 3.7.1.*

(ii) *T^* is an element of . T^* has effective quantifier elimination.*

(iii) *T is an element of . T has effective quantifier elimination. T is complete.*

Proof. (i) It is not difficult to restate and reindex Axioms I–VI to get a sequence Φ as in the statement of Lemma 3.7.1. For example, if k, m, and $s > 0$ are such that $s \in D_k$ and $m \in U_s$, we can combine the appropriate instances of Ax II and IV into a single formula of the form:

$$\neg\left(\bigwedge_{i<n} P_s(x_i)\right) \vee \left(\bigwedge_{i<n} P_{s-1}(x_i) \wedge R^k_\ell(x_0, \ldots, x_{k-1})\right).$$

By Lemma 3.7.1, there is thus a procedure that decides whether a given quantifier-free L-formula ϕ is consistent with Axioms I–VI.

(ii) Follows from part (i) and Lemma 3.7.1. Similar to the proof of Corollary 3.4.2.

(iii) Follows from part (ii).

□

Lemma 3.7.3. *T is consistent.*

Proof. Since T has effective quantifier elimination, there is a procedure to check whether a given L-formula ϕ is consistent with Axioms I–VI. We can use this procedure to decide, given a finite L-structure $\mathcal{F}$ and an s such that $\mathcal{F}$ satisfies Axiom I and $\mathcal{F} \models \neg P_s(a)$ for each element a, whether $\mathcal{F}$ satisfies Axioms I–VI. Hence we can construct an enumeration $\mathbb{K} = \langle \mathcal{F}_0, \ldots \rangle$ of all finite L-structures satisfying Axioms I–VI and having such an s, together with a sequence $\langle s_0, s_1, \ldots \rangle$ where each s_i is the s for the corresponding $\mathcal{F}_i$. Then $\mathbb{K}$ meets the criteria listed in Lemmas 2.6.7 and 2.6.8. It follows that $\mathbb{K}$ has a Fraïssé limit $\mathcal{A} \models T$, $\mathcal{A} \in$. □

We now prove a few results about the types of T.

Lemma 3.7.4. *Let N be the number fixed at the beginning of §3.7.1. Fix $k < N$. There is a 1-type $p_k(x)$ of T with $C_k(x) \in p_k(x)$ and $P_s(x) \in p_k(x)$ for every $s \in M$, and $\neg C_i(x)$ is in $p_k(x)$ for every $i \neq k$.*

Proof. A Fraïssé construction similar to the proof of Lemma 3.7.3. In this case, we allow at most one element a of every $\mathcal{F}$ to have $\mathcal{F} \models P_s(a)$ for all s. □

Lemma 3.7.5. *Recall that the set D_N is infinite by choice of N. There is an N-tuple $\langle p_0(x_0), \ldots,$ $p_{N-1}(x_{N-1})\rangle$ of 1-types such that no N-type extends $p_0(x_0) \cup \cdots \cup p_{N-1}(x_{N-1})$.*

Proof. Let $p_0(x), \ldots, p_{N-1}(x)$ be the nonprincipal 1-types described in Lemma 3.7.4. Consider the tuple $\langle p_0(x_0), \ldots, p_{N-1}(x_{N-1})\rangle \in$. We claim that there is no N-type $q(x_0, \ldots, x_{N-1})$ extending $p_0(x_0) \cup \cdots \cup p_{N-1}(x_{N-1})$. Suppose for a contradiction that such a q does exist. Since whenever $k \neq \ell$ we have $p_k(x)$ containing $C_k(x)$ but $p_\ell(x)$ containing $\neg C_k(x)$, we know that q contains $x_k \neq x_\ell$ for all such k, ℓ. It follows by Ax IV and V that the set $\{s : q \text{ contains } R_s^N(x_0, \ldots, x_{N-1})\}$ is a separating set for $\langle U_s, V_s\rangle_s$, a contradiction. □

Lemma 3.7.6. *T has countably many types.*

Proof. We outline a procedure for enumerating types and argue that the enumeration is exhaustive. Note that, by effective quantifier elimination, it suffices to enumerate the quantifier-free parts of the types.

We use a dovetailing method. For each triple $\langle \ell, m, s \rangle$, we assume that D_ℓ is bounded above by s, and try to list all $(\ell + m)$-types $p(\bar{x}, \bar{y})$, where $\bar{x}$ has length ℓ and $\bar{y}$ has length m, such that p restricted to x_i is a nonprincipal 1-type for each x_i, and $\neg P_s(y_j)$ holds for each y_j. Beginning with P_0 and R_0^1, fill in the atomic diagram of $(\bar{x}, \bar{y})$ relation-by-relation in a way consistent with T. If D_ℓ is indeed bounded above by s, then $\neg R_t^k(\bar{z})$ necessarily holds for all $t > s$ and all $\bar{z}$ taken from $\bar{x}, \bar{y}$, so for relations and R_t^k, P_t with $t > s$, our diagrams are very straightforward. If our assumption was wrong and D_ℓ is not bounded above by s, we will find out, say at stage s_0; for all $t > s_0$ and all z taken from $\bar{x}, \bar{y}$, we let $\neg R_t(z)$ hold. Finally, close the enumeration under all possible renamings of variables.

Now suppose that $q(\bar{z})$ is any type of T. Using bounded Σ_1^0 comprehension to determine which entries of $\bar{z}$, if any, realize a nonprincipal 1-type. We can then find a 1-type $p(\bar{x}, \bar{y})$ of T and a bijection π from the entries of $(\bar{x}, \bar{y})$ to those of $\bar{z}$ such that $\bar{x}$ are the only variables of p whose restriction is a nonprincipal 1-type, and q is exactly $p(\pi(\bar{x}, \bar{y}))$. So q is covered by the enumeration. □

The final lemma of this subsection is used in showing that the types of T have some amalgamation properties—namely, in the special case described in §3.7.3, the 1-point full amalgamation property, and in §3.7.4, the pairwise full amalgamation property.

Lemma 3.7.7. *If $\mathcal{F}$ is a finite model of Axioms I–VI with domain F, then there is a $t \in M$ such that, for all subsets $G \subseteq F$, either:*

- *$D_{|G|}$ is bounded above by t; or*
- *$\mathcal{F} \models \neg P_t(a)$ for some $a \in G$.*

Proof. By $\mathsf{I}\Sigma^0_1$, we may partition F into two sets:

$$F_0 = \{a \in F : \mathcal{F} \models \neg P_s(a) \text{ for some } a\},$$

$$F_1 = \{a \in F : \mathcal{F} \models P_s(a) \text{ for all } a\}.$$

By Σ^0_1 bounding, we may fix an $s_0 \in M$ such that $\mathcal{F} \models \neg P_{s_0}(a)$ for all $a \in F_0$. If F_1 is empty, then s_0 is the desired t. Otherwise, write $F_1 = \{a_0, \ldots, a_{k-1}\}$ without repetition, and consider D_k. If D_k were infinite, then

$$\{s : \mathcal{F} \models R^k_s(a_0, \ldots, a_{k-1}) \text{ and } s \text{ is } k^{\text{th}} \text{ least in } D_k\}$$

would form a separating set for $\langle U_s, V_s \rangle_s$ by Axioms III and IV, a contradiction. Therefore D_k has some upper bound $s_1 \in M$. Now $t = \max(s_0, s_1)$ is as desired. $\square$

3.7.3 The first application

Suppose that $(M,)$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$. Obtain a theory T by performing the construction of §3.7.1 with the following extra constraint on the sequence $D_1 \subseteq D_2 \subseteq \cdots$: There is an N_0 such that D_{N_0} is finite but D_{2N_0} is infinite. To see that this is possible, let $E_1 \subseteq E_2 \subseteq \cdots$ be a sequence witnessing the failure of $\mathsf{I}\Sigma^0_2$ as in Lemma 3.1.9(iii), let N_0 be such that E_{N_0} is infinite, and define $D_k = \emptyset$ for all $k < N_0$, and let $D_{N_0+k} = E_k$ for all $k \in M$. Then the results of the Verification section §3.7.2 apply; let X be a sequence of all types of T.

Lemma 3.7.8. *Suppose that $p(\bar{x})$ is an m-type of T and that $q_0(\bar{x}, y)$ and $q_1(\bar{x}, z)$ are $(m+1)$-types of T extending p. Then there is $t^* \in M$ such that, for any string $\bar{a} = \langle a_0, \ldots, a_{k-1} \rangle$ taken from the elements of $\bar{x}, y$, and z, one of the following holds:*

- *There are distinct i, j such that $(a_i = a_j)$ is in $q_0 \cup q_1$; or*

- *D_k is bounded above by t^*; or*

- *$\bigwedge_{i<k} P_d(a_i)$ is not in $q_0 \cup q_1$, where $d = \min\{s \in D_k : s \geq t^*\}$.*

Proof. Let X be the set of all tuples $\bar{a}$ taken from $\bar{x}$, y, and z such that $i \neq j$ implies that $(a_i = a_j)$ is not in $q_0 \cup q_1$. Form the subset

$$Y = \left\{ \bar{a} \in X : (\exists s \in D_{|a|}) \left[\bigwedge_{i<k} P_s(a_i) \notin q_0 \cup q_1 \right] \right\}. \tag{3.2}$$

By Σ^0_1 bounding, there is $t_0 \in M$ bounding all s needed in equation (3.2). Now let k_0 be the greatest length of any string in the complement $X - Y$. By the pigeonhole principle, there is a substring $\bar{b}$ of $\bar{a}$ of length $k_1 \geq k_0 - 1$ with all entries taken from either $\bar{x}^\frown y$ or $\bar{x}^\frown z$. Let t_0 and t_1 be the values of t given by Lemma 3.7.7 for q_0 and q_1, respectively. Then D_{k_1} is finite with upper bound $t^*_0 = \max(t_0, t_1)$. By construction of $\langle D_0, D_1, \ldots \rangle$, it follows that D_{k_0} is also finite, say with upper bound t^*_1. Then $t^* = \max(t^*_0, t^*_1)$ is the desired t^*. $\square$

Lemma 3.7.9. *X has the 1-point full amalgamation property.*

Proof. Suppose $p(\bar{x})$ is an m-type, and $q_0(\bar{x}, y)$ and $q_1(\bar{x}, z)$ are $(m+1)$-types extending p. Let t^* be the number given by Lemma 3.7.8 for the union $q_0 \cup q_1$. We extend $q_0 \cup q_1$ to a type $r(\bar{x}, \bar{y}, \bar{z})$ in three steps. First, compute U_{t^*} and V_{t^*}, and, using the effective quantifier elimination from Lemma 3.7.2, fill in the atomic formulas $R^k_s(\bar{a})$ for $s < t^*$ in a way consistent with Axioms I–VI. Next, for all $s > t^*$, fill in the remaining atomic formulas as $\neg P_s(a)$ and $\neg R^k_s(\bar{a})$. Lastly, complete the elementary diagram using the effective quantifier elimination given by Lemma 3.7.2. $\square$

Lemma 3.7.10. *X does not have the pairwise free amalgamation property.*

Proof. Recall from the beginning of this subsection that N_0 is a natural number such that D_{N_0} is finite but D_{2N_0} is infinite. Let $\langle p_0, \ldots, p_{2N_0-1} \rangle$ be a sequence of

1-types as described in Lemma 3.7.5. It is straightforward to construct a pair of N_0-types $q_0(x_0, \ldots, x_{N_0-1})$ and $q_1(x_{N_0}, \ldots, x_{2N_0-1})$ extending $p_0(x_0)\cup\cdots\cup p_{N_0-1}(x_{N_0-1})$ and $p_{N_0}(x_{N_0}) \cup \cdots \cup p_{2N_0-1}(x_{2N_0-1})$, respectively. But there is no $2N_0$-type r extending $q_0 \cup q_1$. $\square$

We are ready to prove the following part of Theorem 3.2.23:

Proposition 3.7.11. $\mathsf{RCA}_0 \vdash (\mathsf{1PT\ FULL} \to \mathsf{PW\ FREE}) \to (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2)$.

Proof. We show the contrapositive. Suppose that $(M,)$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$. Let T and X be as described at the beginning of this subsection. By Lemmas 3.7.2, 3.7.3, 3.7.9, and 3.7.10, T is a complete consistent theory and X is an enumeration of all types with the 1-point full amalgamation property but without the pairwise free amalgamation property. $\square$

3.7.4 The second application

Suppose that $(M,)$ is a model of $\mathsf{RCA}_0+\neg\mathsf{WKL}_0+\neg\mathsf{I}\Sigma^0_2$. Again we obtain a theory T by the construction of §3.7.1, this time using a sequence $D_1 \subseteq D_2 \subseteq \cdots$ such that D_n finite implies D_{2n} finite, as in Lemma 3.1.9(iv). Let N be the number fixed in §3.7.1, and recall that D_N is infinite. $X = \langle p_0, p_1, \ldots\rangle$ be a sequence of all types such that, for each $k < N$, p_k is equal to the p_k described in Lemma 3.7.5. (To see this is possible, let X be the sequence of types produced by prepending the list $\langle p_0, \ldots, p_N\rangle$ from Lemma 3.7.5 onto the list of all types given by Lemma 3.7.6.)

Lemma 3.7.12. *Suppose that $p(\bar{x})$ is a type of T and that $q_0(\bar{x}, \bar{y})$ and $q_1(\bar{x}, \bar{z})$ are types of T extending p. Then there is $t^* \in M$ such that, for any string $\bar{a} = \langle a_0, \ldots, a_{k-1}\rangle$ taken from the elements of $\bar{x}, \bar{y}$, and $\bar{z}$, one of the following holds:*

- *There are distinct i, j such that $(a_i = a_j)$ is in $q_0 \cup q_1$; or*

- D_k *is bounded above by* t^**; or*
- $\bigwedge_{i<k} P_d(a_i)$ *is not in* $q_0 \cup q_1$*, where* $d = \min\{s \in D_k : s \geq t^*\}$.

Proof. Similar to the proof of Lemma 3.7.8, except this time the Pigeonhole Principle tells us only that $k_1 \geq k_0/2$. Our more stringent requirement that D_n finite imply D_{2n} finite allows us to get a bound t^* by the same reasoning as before. □

Lemma 3.7.13. *X has the pairwise full amalgamation property.*

Proof. Similar to the proof of Lemma 3.7.9, using Lemma 3.7.12 in place of 3.7.8. □

Lemma 3.7.14. *X does not have the finite free amalgamation property.*

Proof. By choice of the initial segment $\langle p_0(x_0), \ldots, p_{N-1}(x_{N-1})\rangle$ and Lemma 3.7.5. □

We are ready to prove the remaining part of Theorem 3.2.23.

Proposition 3.7.15. $\mathsf{RCA}_0 \vdash (\mathsf{PW\ FULL} \rightarrow \mathsf{FIN\ FREE}) \rightarrow (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2)$.

Proof. We show the contrapositive. Suppose that $(M,)$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$, and let T, X be as specified at the beginning of this subsection. Then by Lemmas 3.7.2, 3.7.3, 3.7.13, and 3.7.14, we know X is a sequence of all types of a complete consistent theory, and X has the pairwise full but not the finite free amalgamation property. □

We now prove a simple lemma, and proceed to the final part of Theorem 3.2.14.

Lemma 3.7.16. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash$ (*If a complete consistent theory has an $\emptyset$-saturated model, then every enumeration of all its types has the finite free amalgamation property*).

Proof. Suppose that T^* is a complete consistent theory, $\mathcal{A}$ is an $\emptyset$-saturated model, $X^* = \langle p_0^*, \ldots \rangle$ is a sequence of all types, and $\langle i_0, \ldots, i_{n-1} \rangle$ is a tuple of indices. Each $p_{i_k}^* = p_{i_k}^*(\bar{x}_k)$ is realized by some tuple $\bar{a}$. By the characterization of $\mathsf{B}\Sigma_2^0$ found in Lemma 3.1.11, we may form a tuple $\langle \bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}} \rangle$ of tuples such that $p_{i_k}^*(\bar{a}_{j_k})$ holds for each $k < n$. Then the type $\mathsf{tp}^{\mathcal{A}}(\bar{a}_{j_0}{}^\frown \cdots {}^\frown \bar{a}_{j_{n-1}})$ extends every $p_{i_k}^*(\bar{x}_k)$, as required. $\square$

Proposition 3.7.17. $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2^0 \vdash$ (*If a complete consistent theory has a sequence of all types with the pairwise full amalgamation property, then it has an $\emptyset$-saturated model*) $\rightarrow (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma_2^0)$.

Proof. Immediate from Proposition 3.7.15 and Lemma 3.7.16. $\square$

CHAPTER 4

WEAK TRUTH TABLE DEGREES OF MODELS

This chapter was published in the Notre Dame Journal of Formal Logic in 2015 [5].

4.1 Introduction

A *first-order relational structure*, henceforth simply a *structure*, is a tuple of the form $\mathcal{A} = \big(A, (R_k^{\mathcal{A}})_{k\in I}\big)$, where A is a nonempty set (called the *universe* of $\mathcal{A}$), I is some set used for indexing, and each $R_k^{\mathcal{A}}$ is a set of tuples from A of a common arity $\mathsf{ar}(R_k)$—that is, $R_k^{\mathcal{A}} \subseteq A^{\mathsf{ar}(R_k)}$. We are interested in those $\mathcal{A}$ for which the universe A is ω and the indexing set I is either ω or a finite set. Unless otherwise specified, we assume that $I = \omega$. We also assume that the sequence $(\mathsf{ar}(R_0), \mathsf{ar}(R_1), \ldots)$, called the *signature* of $\mathcal{A}$, is computable. By padding with empty relations if necessary, we make the assumption (convenient in some calculations below) that $\mathsf{ar}(R_k) \leq k/2$ for all k. When I is a finite set, we say that $\mathcal{A}$ has *finite signature.*

We are interested in the computational content of a structure $\mathcal{A}$. To give this a more precise meaning, we identify $\mathcal{A}$ with its *atomic diagram* $\mathcal{D}\mathcal{A} = \{\langle k, \vec{u}\rangle : \vec{u} \in R_k^{\mathcal{A}}\}$. Since this $\mathcal{D}\mathcal{A}$ is a set of natural numbers, it can be assigned a degree of complexity in the usual computability-theoretic sense. Recall that a *reducibility* is reflexive, transitive, binary relation $\leq_r$ on 2^ω. Such a $\leq_r$ induces an equivalence relation $\equiv_r$ on 2^ω, by $A \equiv_r B \iff [A \leq_r B$ and $B \leq_r A]$. We let $(\mathcal{D}_r, \leq)$ denote the partially ordered structure whose universe is the set of all $\equiv_r$-equivalence classes, and whose order is induced by $\leq_r$. The elements of $\mathcal{D}_r$ are called *r-degrees.* A structure $\mathcal{A}$ is said to have r-degree $\deg_r(\mathcal{A})$, where

$$\deg_r(\mathcal{A}) = \deg_r(D(\mathcal{A})) = \{B \subseteq \omega : B \equiv_r D(\mathcal{A})\}.$$

In most cases, $\deg_r(\mathcal{A})$ is not invariant under isomorphism—that is, if $\mathcal{B}$ is an isomorphic copy of $\mathcal{A}$, it is possible that $\deg_r(\mathcal{B}) \neq \deg_r(\mathcal{A})$. Define the *$r$-degree spectrum* of $\mathcal{A}$ to be:

$$\mathsf{spec}_r\mathcal{A} = \{\mathbf{b} \in \mathcal{D}_r : (\exists \mathcal{B} \cong \mathcal{A})[\deg_r(\mathcal{B}) = \mathbf{b}]\}.$$

In this paper, we concentrate our attention on the cases where $\leq_r$ is either *Turing reducibility* ($\leq_T$) or *weak truth table reducibility* ($\leq_{wtt}$). *Truth table reducibility* ($\leq_{tt}$) also appears. We assume some familiarity with $\leq_T$, $\leq_{wtt}$, and $\leq_{tt}$, and anchor our notation to texts such as Lerman [42] and Soare [64]. Considerable effort has already gone into studying $\mathsf{spec}_T\mathcal{A}$, and, recently, authors have begun studying other sorts of degree spectrum. For example, Soskov and Soskova [67, 68] have examined the enumeration degree spectrum $\mathsf{spec}_e\mathcal{A}$, and Greenberg–Knight [22] have lifted the Turing degree spectrum into the setting of higher recursion theory. Chisholm et al. [8] recently examined the tt and wtt *degree spectrum of a relation*—a notion distinct from, but related to, the *degree spectrum* studied here.

Although our new results concern the wtt degree spectrum, we draw inspiration from, and analogies with, the past few decades' research on $\mathsf{spec}_T\mathcal{A}$. The reader can find much more information on $\mathsf{spec}_T\mathcal{A}$ gathered in the text of Ash and Knight [2] and in the shorter survey article of Knight [38].

We begin in §4.1.1 with a discussion of some known theorems about $\mathsf{spec}_T\mathcal{A}$, and their relation to our new results about $\mathsf{spec}_{wtt}\mathcal{A}$. In §§4.2, 4.3, 4.5, and 4.6, we look at these new results and their proofs. The longest of these proofs, that of Theorem 4.3.6, comprises §4.4.

4.1.1 Background and overview

We begin with a brief overview of our new results, together with the questions and the known theorems—mainly about the Turing degree spectrum—that inspired them. We hope that this will, in one swoop, motivate and expose the work in the rest of the paper. Most of the results in this section are stated in a simplified or weakened form in order to emphasize the main idea over the details. In each case we indicate where, in the sections that follow, to find the stronger version and its proof.

For a fixed reducibility $\leq_r$, our questions about r-degree spectra fall into one or more of the following broad classes.

Main questions. I. Given a particular structure $\mathcal{A}$, what can we say about $\mathsf{spec}_r\mathcal{A}$?
II. Given a particular class of structures (for example, the models of some fixed theory), what can we say about their r-degree spectra?
III. Given a class $\mathcal{C} \subseteq 2^\omega$ of reals, is it possible to write $\mathcal{C} = \bigcup \mathsf{spec}_r\mathcal{A}$ for some structure $\mathcal{A}$? If so, what more can we say of such an $\mathcal{A}$?

Questions of the third variety give a useful point of comparison between the Turing and wtt degree spectra, and between these and other methods of defining a class of reals. (For instance, given a structure $\mathcal{A}$, the collection $\bigcup \mathsf{spec}_{wtt}\mathcal{A}$ is always a $\Sigma_1^{1,\mathcal{A}}$ class.) A good first step in our study of the wtt degree spectrum is to check that it is not the same object as the Turing degree spectrum. In fact, except for some trivial cases, there are strictly more classes of reals that can be defined by a wtt degree spectrum than by a Turing degree spectrum.

Theorem 4.1.1. *(i) If $\mathcal{A}$ is a structure, then either $\mathsf{spec}_T\mathcal{A}$ consists of a single Turing degree, or there is a structure $\mathcal{B}$ such that $\mathsf{spec}_{wtt}\mathcal{B}$ coincides with $\mathsf{spec}_T\mathcal{A}$ in the sense that $\bigcup \mathsf{spec}_{wtt}\mathcal{B} = \bigcup \mathsf{spec}_T\mathcal{A}$. In fact, we may take $\mathcal{B}$ to be a graph.*
(ii) There is a structure $\mathcal{A}$ with finite signature such that $\mathsf{spec}_T\mathcal{A}$ is not a singleton, and $\bigcup \mathsf{spec}_{wtt}\mathcal{A} \neq \bigcup \mathsf{spec}_T\mathcal{A}$.

(iii) There is a structure $\mathcal{A}$ with finite signature such that $\mathsf{spec}_T\mathcal{A}$ is not a singleton, and $\bigcup \mathsf{spec}_{wtt}\mathcal{A} \neq \bigcup \mathsf{spec}_T\mathcal{B}$ for any structure $\mathcal{B}$.

Parts (i) and (ii) are immediate from Propositions 4.6.3 and 4.6.5 below. Part (iii) can be deduced from Part (ii) and Theorem 4.1.3 below. The next step is to ask for a characterisation of the wtt degree spectra which coincide with a Turing degree spectrum. It can be more intuitive to frame such questions in terms of classes of degrees, rather than of reals. We make frequent use of the following definitions.

Definition 4.1.2. Let $\mathcal{C} \subseteq \mathcal{D}_r$ be a class of r-degrees, and fix a degree $\mathbf{a} \in \mathcal{D}_r$. Write $\mathcal{D}_r(\geq \mathbf{a}) = \{\mathbf{b} \in \mathcal{D}_r : \mathbf{b} \geq \mathbf{a}\}$. We say that $\mathcal{C}$ *contains the cone above* $\mathbf{a}$ if $\mathcal{D}_r(\geq \mathbf{a}) \subseteq \mathcal{C}$. We say, on the other hand, that $\mathcal{C}$ *avoids the cone above* $\mathbf{a}$ if $\mathcal{D}_r(\geq \mathbf{a}) \cap \mathcal{C} = \emptyset$. A nonempty class $\mathcal{C} \subseteq \mathcal{D}_r$ of r-degrees is called *upward closed* if, for any degree $\mathbf{a} \in \mathcal{C}$, the class $\mathcal{C}$ contains the cone above $\mathbf{a}$.

The following dichotomy theorem was proved by Knight [37].

Theorem 4.1.3 (Knight)**.** *Let $\mathcal{A}$ be any structure. Either $\mathsf{spec}_T\mathcal{A}$ is upward closed, or $\mathsf{spec}_T\mathcal{A}$ is a singleton.*

We give the original, more detailed formulation, along with a sketch of a proof, below, as Theorem 4.2.2. As we shall see, $\mathsf{spec}_{wtt}\mathcal{A}$ is a singleton if and only if $\mathsf{spec}_T\mathcal{A}$ is a singleton; as a consequence, any wtt degree spectrum that coincides with a Turing degree spectrum is itself upward closed. We now present a new dichotomy for $\mathsf{spec}_{wtt}\mathcal{A}$, similar to Theorem 4.1.3, which gives a necessary condition for the wtt degree spectrum to be upward closed.

Theorem 4.1.4. *Let $\mathcal{A}$ be any structure. Either $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above some degree $\mathbf{a}$, or $\mathsf{spec}_{wtt}\mathcal{A}$ avoids the cone above some degree $\mathbf{a}$.*

Note that only one of the two alternatives in Theorem 4.1.4 can hold, since any two degrees $\mathbf{a}_1, \mathbf{a}_2$ have a common upper bound in the wtt degrees—namely their join $\mathbf{a}_1 \vee \mathbf{a}_2$. Note also that, although Theorem 4.1.4 could easily be deduced from certain large cardinal hypotheses[1], we actually prove a stronger result by specifying a bound on $\mathbf{a}$ (Theorem 4.3.6 and Corollary 4.3.7 below) within ZFC.

In §4.3 below we construct a structure $\mathcal{A}$ such that $\mathsf{spec}_{wtt}\mathcal{A}$ avoids a cone but is not a singleton. This shows that Theorem 4.1.4 cannot, without some extra conditions, be extended to a perfect analogue of Theorem 4.1.3. We now suggest some candidate conditions:

Question 4.1.5. (a) Is it the case that, if $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed, then the union $\bigcup \mathsf{spec}_{wtt}\mathcal{A}$ is equal to $\bigcup \mathsf{spec}_T\mathcal{B}$ for some $\mathcal{B}$? (b) Is it the case that, if $\mathsf{spec}_{wtt}\mathcal{A}$ contains a cone, then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed?

We answer question (a) in the negative. In fact, it is easy to see from the proof of Proposition 4.6.5 below that the $\mathsf{spec}_{wtt}\mathcal{A}$ of Theorem 4.1.1(ii) and (iii) is upward closed. Although we do not have a full answer to question (b), we do succeed in finding *examples* of a structure $\mathcal{A}$ for which $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed. In §4.5 we list some additional conditions on a structure $\mathcal{A}$ give an affirmative answer to questions (a) and (b) for that $\mathcal{A}$.

Here is another remarkable limitation on the Turing degree spectrum, essentially proved in Knight [37].

Theorem 4.1.6 (Knight)**.** *Suppose $\mathcal{A}$ is a structure, $(\mathbf{e}_n)_{n\in\omega}$ is a sequence of Turing degrees, and $\mathsf{spec}_T\mathcal{A} \subseteq \bigcup_{n\in\omega} \mathcal{D}_T(\geq \mathbf{e}_n)$. Then there is an $n_0 \in \omega$ such that $\mathsf{spec}_T\mathcal{A} \subseteq \mathcal{D}_T(\geq \mathbf{e}_{n_0})$.*

[1] Namely, if $a^\sharp$ exists for all reals a, then a wtt version of Martin's Cone Lemma [44] gives the desired cones.

One of our new theorems, proved in §4.5 below, gives a similar-looking result for wtt degree spectra of structures with finite signature.

Theorem 4.1.7. *Suppose $\mathcal{A}$ is a structure with finite signature, $(\mathbf{e}_n)_{n\in\omega}$ is a sequence of wtt degrees, and $\mathsf{spec}_{wtt}\mathcal{A} \subseteq \bigcup_{n\in\omega} \mathcal{D}_{wtt}(\geq \mathbf{e}_n)$. Then there is an $n_0 \in \omega$ such that $\mathbf{e}_{n_0} = \mathbf{0}$.*

The most direct analogue of Theorem 4.1.7 does not hold in the Turing case; for example, an early paper of Richter [55] constructs, for each Turing degree $\mathbf{a} > \mathbf{0}$, a partially-ordered set $P = (\omega, \preceq)$ such that $\mathsf{spec}_T P = \mathcal{D}_T(\geq \mathbf{a})$.

Another known result is that every nonsingleton Turing degree spectrum is the Turing degree spectrum of a graph. A highly effective construction can be found in the paper of Hirschfeldt–Khoussainov–Shore–Slinko [27].

Theorem 4.1.8 (H–K–S–S)**.** *If $\mathcal{A}$ is a structure and $\mathsf{spec}_T\mathcal{A}$ is not a singleton, then there is a graph $G = (\omega, E^G)$ such that $\mathsf{spec}_T G = \mathsf{spec}_T\mathcal{A}$.*

Deliberately ignoring the singleton case, we say that the theory of graphs is *universal for Turing degree spectra*. One might ask whether the theory of graphs is similarly universal for wtt degree spectra. Sadly, it is not. We can see this by taking a structure $\mathcal{B}$ and a wtt degree $\mathbf{a} > \mathbf{0}$ such that $\mathsf{spec}_{wtt}\mathcal{B} \subseteq \mathcal{D}_{wtt}(\geq \mathbf{a})$ (a suitable $\mathcal{B}$ is constructed in Proposition 4.6.2 below), and invoking Theorem 4.1.7 with $\mathbf{e}_n = \mathbf{a}$ for all n. We leave open the question of whether a suitable analogue can be found when we consider only structures with finite signature.

Question 4.1.9. Is there a fixed, finite $n \in \omega$ such that, if $\mathcal{A}$ is a structure with finite signature, then there is a structure $\mathcal{B}$ on alphabet $(R_0, \ldots, R_{n-1})$ such that $\mathsf{spec}_{wtt}\mathcal{B} = \mathsf{spec}_{wtt}\mathcal{A}$?

4.2 Knight's dichotomy for Turing degree spectra

We have already mentioned, as Theorem 4.1.3, a result of Knight stating that, for a structure $\mathcal{A}$, the spectrum $\mathsf{spec}_T\mathcal{A}$ is either a singleton or upward closed. Because it motivates our definitions and results in §4.3, we now give a more detailed formulation, as Theorem 4.2.2; and because it serves as a prototype for the proofs of Lemmas 4.4.1 and 4.4.2, we also sketch a proof. The following definitions will be used frequently.

Notation. (i) We use the word *permutation* to mean a bijection from ω to ω.

(ii) Given a set S and a permutation π, we say that π *fixes* S if $\pi \upharpoonright S = id_S$.

(iii) Given a permutation π and structures $\mathcal{A}, \mathcal{B}$, we write $\pi : \mathcal{A} \cong \mathcal{B}$ to mean that π is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

(iv) Given numbers $x, z \in \omega$, we write $[x, z)$ to denote the interval $\{y \in \omega : x \leq y < z\}$. We write $[x, \infty)$ to denote the set $\{y \in \omega : x \leq y\}$. Following the usual convention, each natural number $x \in \omega$ is identified with the interval $[0, x)$.

Definition 4.2.1. A structure $\mathcal{A}$ is called *trivial* if there exists a finite set $S \subset \omega$ such that any permutation π fixing S is an automorphism of $\mathcal{A}$. We say that S *witnesses the triviality of* $\mathcal{A}$.

For example, any graph (ω, E) with only finitely (or cofinitely) many edges in E is trivial. A linear order $(\omega, \preceq)$, on the other hand, is never trivial. To see this, given any finite nonempty set S, choose two distinct elements $a, b \notin S$; then the permutation which transposes a and b and fixes all other elements is not an automorphism of $(\omega, \preceq)$.

If S is a finite set witnessing the triviality of a structure $\mathcal{A}$, π is a permutation, and $\mathcal{B}$ is the isomorphic copy of $\mathcal{A}$ given by $\pi : \mathcal{A} \cong \mathcal{B}$, then we can compute the atomic diagram of $\mathcal{B}$ using that of $\mathcal{A}$ and the restricted map $\pi \upharpoonright S$. Since $\pi \upharpoonright S$ is a finite set, this implies that $\mathcal{B} \leq_T \mathcal{A}$; a symmetric argument also gives $\mathcal{A} \leq_T \mathcal{B}$. A trivial structure therefore has only a single degree in its Turing degree spectrum. In particular, it is easy to see that any trivial structure with finite signature has $\{\mathbf{0}\}$ as its Turing degree spectrum.

On the other hand, suppose $\mathcal{A}$ is *not* trivial. Then we can list (noneffectively) an infinite collection of pairs $\{\{a_i, b_i\}\}_i$, pairwise disjoint, where the transposition of any $\{a_i, b_i\}$ is not an automorphism of $\mathcal{A}$. By transposing simultaneously any nonempty subcollection of these pairs $\{a_i, b_i\}$, we again get a permutation which is not an automorphism of $\mathcal{A}$. Thus there are $2^{\aleph_0}$-many different atomic diagrams of structures isomorphic to $\mathcal{A}$. By the pigeonhole priciple, the degree spectrum $\mathsf{spec}_T\mathcal{A}$ has cardinality $2^{\aleph_0}$ as well.

Therefore, no Turing degree spectrum can have cardinality strictly between 1 and $2^{\aleph_0}$: in classifying structures into the *trivial* and the *not trivial*, we uncover a significant gap among the possible Turing degree spectra. The gap is actually much wider, however, as Knight showed in [37].

Theorem 4.2.2 (Knight)**.** *If $\mathcal{A}$ is a structure, then*

(1) $\mathcal{A}$ is not trivial if and only if $\mathsf{spec}_T\mathcal{A}$ is upward closed in the Turing degrees;

(2) $\mathcal{A}$ is trivial if and only if $\mathsf{spec}_T\mathcal{A}$ is a singleton.

We sketch a proof; for a detailed version, the reader should refer to [37].

Definition 4.2.3. If $\mathcal{A}$ is a structure and $X, Y \subseteq \omega$ are sets of natural numbers, then we define the *restricted diagram* $\mathcal{A} \upharpoonright_Y^X$ to be the restriction of $\mathcal{DA}$ to those relations indexed by X and those elements in Y, that is,

$$\mathcal{A} \upharpoonright_Y^X (\langle k, \vec{u} \rangle) = \begin{cases} \mathcal{DA}(\langle k, \vec{u} \rangle) \text{ if } k \in X \text{ and } u_i \in Y \text{ for each } i \\ \uparrow \text{ otherwise.} \end{cases}$$

This $\mathcal{A}\restriction_Y^X$ is seen as a structure with universe Y and alphabet $\{R_i : i \in X\}$. In practice, X and Y will usually be initial segments of ω. When X contains all of $\mathcal{A}$'s relations, we sometimes write $\mathcal{A}\restriction_Y$ for $\mathcal{A}\restriction_Y^X$.

Proof of 4.2.2 (sketch). We have already established Part (2) and the 'if' direction of Part (1) through our discussion of the cardinality of $\mathsf{spec}_T\mathcal{A}$.

We now show the 'only if' direction of Part (1). Suppose that $\mathcal{A}$ is not trivial, and fix any set $C \in 2^\omega$ such that $C \geq_T \mathcal{A}$. We exhibit a permutation π such that, if $\mathcal{B}$ is the unique structure with $\pi : \mathcal{A} \cong \mathcal{B}$, then $\mathcal{B} \equiv_T C$. We get $C \leq_T \mathcal{B}$ by coding the elements of C directly into $\mathcal{B}$; to ensure that $C \geq_T \mathcal{B}$, we build π effectively in C and use the fact that $\mathcal{B} \leq_T \mathcal{A} \oplus \pi$.

Construction. The permutation π is built computably in C as the pointwise limit of a sequence $(\pi_s)_s$ of permutations, alongside which we build a sequence $(m_s)_s$ of natural numbers to act as restraints. Begin with $\pi_0 = id_\omega$ and $m_s = 0$.

At each stage s, suppose that we have already defined π_s and m_s, and that $\mathcal{B}_s$ is the unique structure such that $\pi_s : \mathcal{A} \cong \mathcal{B}_s$. Because $\mathcal{A}$ is not trivial, there is a permutation ρ which fixes the interval $[0, m_s)$ and which is not an automorphism of $\mathcal{B}_s$. In fact, it is easy to see that there is such a ρ fixing $[0, m_s) \cup [m_s + k, \infty)$ for some k. From here it is easy to see that there is a ρ fixing $[0, m_s) \cup [m_s + k, \infty)$ which is not an automorphism of $\mathcal{B}_s\restriction_{m_s+k}^k$; choose the least such k.

Make a list $(G_0, G_1, \ldots, G_{n-1})$ of all possible images of $\mathcal{B}_s\restriction_{m_s+k}^k$ under a permutation of $[0, m_s + k)$ fixing $[0, m_s)$. Find the least $k^* \in \omega$ such that there exist $i, j < n$ with $G_i\restriction_{m_s+k^*}^{k^*}$ and $G_j\restriction_{m_s+k^*}^{k^*}$ unequal, but isomorphic through a permutation fixing $[0, m_s) \cup [m_s + k^*, \infty)$.

Using some fixed computable enumeration of ordered pairs of finite atomic diagrams, choose i, j as above with $\langle G_i\restriction_{m_s+k^*}^{k^*}, G_j\restriction_{m_s+k^*}^{k^*}\rangle$ coming as early as possible in the enumeration. There exist permutations ρ_0, ρ_1, each fixing $[0, m_s) \cup [m_s + k, \infty)$, such that $\rho_0 : \mathcal{B}_s \upharpoonright^k_{m_s+k} \cong G_i$ and $\rho_1 : \mathcal{B}_s \upharpoonright^k_{m_s+k} \cong G_j$. If $s \notin C$, let $\tau = \rho_0 \circ \pi_s$; if $s \in C$, let $\tau = \rho_1 \circ \pi_s$. Find the least $x \in \omega$ such that $\tau(x) \geq m_s + k^*$, and let $y = \tau(x)$. Let σ_s be the permutation which transposes y and $m_s + k^*$, and fixes all other elements. Define the next π_{s+1} by $\pi_{s+1} = \sigma_s \circ \tau$, and define $m_{s+1} = m_s + k^* + 1$. This completes the construction.

Verification. Because at each stage s the functions ρ_0, ρ_1 are permutations fixing $[0, m_s)$ and the bounds $(m_s)_s$ form an increasing sequence, the limit π is an injective partial function from ω into ω. The final transposition $(y, m_s + k^*)$ at each stage guarantees that π is total and surjective. Hence π is a permutation.

Let $\mathcal{B}$ be the unique structure such that $\pi : \mathcal{A} \cong \mathcal{B}$. Using knowledge of $\mathcal{B}$, we can recover the sequence $(m_s)_s$ and the set C inductively, as follows. Suppose that $(m_0, m_1, \ldots, m_s)$ are already known. Find the least $k^* \in \omega$ such that there is a permutation fixing $[0, m_s) \cup [m_s + k^*, \infty)$ which is not an automorphism of $\mathcal{B} \upharpoonright^{k^*}_{m_s+k^*}$. This k^* is the same as the k^* from stage s of the construction. So we may compute $m_{s+1} = m_s + k^* + 1$.

Enumerate all possible images $(H_0, \ldots, H_n)$ of $\mathcal{B} \upharpoonright^{k^*}_{m_s+k^*}$ under a permutation fixing $[0, m_s) \cup [m_s + k^*, \infty)$, and, within the same fixed computable enumeration as before, choose the earliest pair $\langle H_i, H_j \rangle$ with $H_i \neq H_j$. Then $\mathcal{B} \upharpoonright^{k^*}_{m_s+k^*}$ is equal either to H_i, in which case $s \notin C$, or to H_j, in which case $s \in C$.

$\square$

This result is nice enough, and the construction effective enough, that one might wish to adapt it to the wtt case. As we have stated in §4.1.1, the most direct possible analogue—swapping wtt for T in the statement of the theorem—does not hold. Still, the ideas used in proving Theorem 4.2.2 are useful in the wtt case. We come back to this construction in proving Proposition 4.3.4 and Lemma 4.4.2 below.

4.3 A Dichotomy for the wtt degree spectrum

What follows will require notation from computability theory. To streamline the discussion, we fix an enumeration $(\varphi_e)_e$ of some (but not all) computable functions, and we introduce a nonstandard symbol $\hat{\Phi}_e$.

Definition 4.3.1. (i) We let $(\Phi_e)_e$ be the standard effective listing of computable functionals.

(ii) We are interested in those partial computable functions ψ with domain an inital segment of ω, and which are increasing on their domain. We let $(\varphi_e)_e$ be an effective listing of all such ψ.

(iii) Define the *sequence of all wtt-functionals* $(\hat{\Phi}_e)_e$ *operating on structures* as follows. Recall that we identify a structure $\mathcal{A}$ with its atomic diagram $\mathcal{DA} \subseteq \omega$. Given $\mathcal{A}$ and natural numbers $x, s \in \omega$, if $\varphi_{e,s}(x) \downarrow$ and $\Phi^{\mathcal{A}}_{e,s}(x) \downarrow$ while using queries only to $\mathcal{A} \restriction^{\varphi_e(x)}_{\varphi_e(x)}$—that is, asking only oracle questions of the form '$\langle k, y_0, \dots, y_n \rangle \in \mathcal{DA}$?' with each $k, x_i < \varphi_{e,s}(x)$, then $\hat{\Phi}^{\mathcal{A}}_{e,s}(x) \downarrow = \Phi^{\mathcal{A}}_e(x)$. Otherwise, $\hat{\Phi}^{\mathcal{A}}_{e,s}(x) \uparrow$. If there is an s such that $\hat{\Phi}^{\mathcal{A}}_{e,s}(x) \downarrow = y$, then we write $\hat{\Phi}^{\mathcal{A}}_e(x) = y$. Otherwise, we write $\hat{\Phi}^{\mathcal{A}}_e(x) \uparrow$. If $\hat{\Phi}^{\mathcal{A}}_e(x) \downarrow \in \{0, 1\}$ for every $x \in \omega$, then we identify $\hat{\Phi}^{\mathcal{A}}_e$ with a subset of ω in the usual way.

An application of the s-m-n theorem shows that, for any X and $\mathcal{A}$, we have $X \leq_{wtt} \mathcal{A}$ if and only if there is an e such that $X = \hat{\Phi}^{\mathcal{A}}_e$.

Now let us try to determine where the proof of Theorem 4.2.2 breaks down when we substitute $\leq_{wtt}$ for $\leq_T$. The cardinality argument for part (2) carries over unchanged:

Proposition 4.3.2. *A structure $\mathcal{A}$ is trivial if and only if $\mathsf{spec}_{wtt}\mathcal{A}$ is a singleton.*

The construction for the 'only if' direction of Theorem 4.2.2(1) does not on its face give $\mathcal{B} \leq_{wtt} C$, since there might not be a computable bound on the length of the searches used in choosing k. As well, we might not end up with $C \leq_{wtt} \mathcal{B}$, since the sequence $(m_s)_s$, and hence the length of the searches used to compute C, might not have a computable bound.

We can do away with these objections in certain cases. If $\mathcal{A}$ has finite signature, for instance, then surely $C \leq_{wtt} \mathcal{B}$. If $\mathcal{A} = (\omega, \leq^{\mathcal{A}})$ is a linear order, then at each stage s of the construction we get $m_{s+1} \leq m_s + 2$, giving $\mathcal{B} \leq_{wtt} C$. Hence $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed for any linear order $\mathcal{A}$. We examine the finite-signature case more closely in §4.5.

It is also useful to consider degree-theoretic conditions on $\mathcal{A}$.

Definition 4.3.3. We say that a set $A \in 2^\omega$ is *of* $\mathbf{0}$*-dominated degree* (also called *of hyperimmune-free degree*) if, for every total function $f \leq_T A$, there is a total computable function g such that $(\forall x)[f(x) \leq g(x)]$. Equivalently, we could replace '$f \leq_T A$' in this definition with '$\mathsf{graph}(f) \leq_{wtt} A$,' where $\mathsf{graph}(f) = \{\langle x, y\rangle : y = f(x)\}$.

From our point of view, structures of $\mathbf{0}$-dominated degree behave nicely.

Proposition 4.3.4. *If $\mathcal{A}$ is not trivial and is of $\mathbf{0}$-dominated degree, then $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above* $\deg_{wtt}(\mathcal{A})$*. In particular, if $\mathcal{A}$ is computable and not trivial, then $\mathsf{spec}_{wtt}\mathcal{A}$ is all of $\mathcal{D}_{wtt}$.*

Proof. Suppose that $\mathcal{A}$ is of $\mathbf{0}$-dominated degree, and fix any set $C \geq_{wtt} \mathcal{A}$. Build $\mathcal{B} \equiv_T C$ using the construction for Theorem 4.2.2. We use this construction to define two functions f and g. Let f be given by $f(s) = m_s$, and let $g(s) = m_s + \ell$, where ℓ is the greatest among all k used in steps $t \leq s$ of the construction. Then $g \leq_T \mathcal{A}$, so there is a total computable function ψ such that $(\forall x)[g(x) \leq \psi(x)]$. Note that f is dominated by ψ in the same way.

In building $\mathcal{B}\restriction^{\omega}_{m_s}$ from $\mathcal{A} \oplus C$, we use only queries to $C \restriction s+1$ and to $\mathcal{A}\restriction^{m_s+k}_{m_s+k}$. Since and $s+1$ and $m_s + k$ are no greater than $\psi(s)$, this means $\mathcal{B} \leq_{wtt} C$. On the other hand, in recovering $C(s)$ from $\mathcal{B}$, we use only queries to $\mathcal{B}\restriction^{m_{s+1}}_{m_{s+1}}$. Since m_{s+1} is no greater than $\psi(s)$, this implies $C \leq_{wtt} \mathcal{B}$, and hence $\mathcal{B} \equiv_{wtt} C$. $\square$

One last approach is to consider a bounded version of triviality for structures. Recall from Definition 4.2.1 the notion of a finite set *witnessing the triviality* of a structure.

Definition 4.3.5. A structure $\mathcal{A}$ is *w-trivial* if for each total computable function f there is a finite set S witnessing the triviality of the reduct $\mathcal{A}\restriction^{f(|S|)}_{\omega}$.

It is immediate from the definitions that any trivial structure is also w-trivial. There do, however, exist structures which are w-trivial but not trivial. An easy example can be found in §4.6 below.

A structure $\mathcal{A}$ that is w-trivial but not trivial must have $\mathsf{spec}_{wtt}\mathcal{A}$ of size $2^{\aleph_0}$. Such a $\mathsf{spec}_{wtt}\mathcal{A}$ is nonetheless far from upward-closed within the wtt degrees, to the extent that there is a set X such that $\mathsf{spec}_{wtt}\mathcal{A}$ avoids the cone above $\deg_{wtt}(X)$. In fact, we shall exhibit a whole family of such X in the form of a relativised $\Pi^{0,\mathcal{A}}_1$ class. A structure that is not w-trivial, on the other hand, is amenable to a version of the proof of Theorem 4.2.2, which will be enough to show that its wtt degree spectrum does contain *some* upward cone. What we have stated is the following theorem.

Theorem 4.3.6. *Given a structure $\mathcal{A}$:*

(1) If $\mathcal{A}$ is not w-trivial then there is a set $B \leq_T \mathcal{A}$ such that $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above $\deg_{wtt}(B)$.

(2) If $\mathcal{A}$ is w-trivial then there is a nonempty relativised $\Pi^{0,\mathcal{A}}_1$ class $P \subseteq 2^\omega$ such that $\mathsf{spec}_{wtt}\mathcal{A}$ avoids the cone above $\deg_{wtt}(X)$ for every $X \in P$.

See §4.4 for a proof of this theorem. Again, there cannot be wtt degrees $\mathbf{a}, \mathbf{b}$ such that $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above $\mathbf{a}$ and avoids the cone above $\mathbf{b}$, since the intersection $\mathcal{D}_{wtt}(\geq \mathbf{a}) \cap \mathcal{D}_{wtt}(\geq \mathbf{b})$ is nonempty. Hence our classification of structures into the *w-trivial* and the *not w-trivial* admits a simple degree-theoretic characterisation—namely, the dichotomy of Theorem 4.1.4. With some additional effort, we can get a localised version:

Corollary 4.3.7. *Given a structure $\mathcal{A}$:*

(1) $\mathcal{A}$ is not w-trivial if and only if there is a set $C \geq_{wtt} \mathcal{A}$, $C \equiv_T \mathcal{A}$, such that $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above $\deg_{wtt}(C)$.

(2) $\mathcal{A}$ is w-trivial if and only if there is a set $C \geq_{wtt} \mathcal{A}$, $C' \leq_{tt} \mathcal{A}'$, such that $\mathsf{spec}_{wtt}\mathcal{A}$ avoids the cone above $\deg_{wtt}(C)$. (Here $\mathcal{A}'$ is the Turing jump of the atomic diagram of $\mathcal{A}$.)

The proof will use the following relativised, truth-table version of the Low Basis Theorem of Jockusch–Soare [32].

Lemma 4.3.8. *Let A be a set of natural numbers. If P is a nonempty $\Pi^{0,A}_1$ class, then there is an element $X \in P$ such that $X' \leq_{tt} A'$.*

The proof of this lemma, omitted here, is a straightforward relativisation of the proof of the Superlow Basis Theorem due to Marcus Schaefer—see, for example, Downey and Hirschfeldt [12, Theorem 2.19.9].

Proof of Corollary 4.3.7. For (1), take B as in Theorem 4.3.6 and let $C = \mathcal{A} \oplus B$.

For (2), take P as in Theorem 4.3.6 and let $Q = \{\mathcal{A} \oplus Y : Y \in P\}$. Then Q is a nonempty $\Pi^{0,\mathcal{A}}_1$ class, and $X \in Q$ implies $\mathcal{A} \leq_{wtt} X$. Apply the Lemma to Q. □

Note that it is not possible to replace $C' \leq_{tt} \mathcal{A}'$ in Corollary 4.3.7(2) with the stronger condition $C \equiv_T \mathcal{A}$. For, if $\deg_T(\mathcal{A})$ is not $\mathbf{0}$-dominated and consists of exactly one wtt-degree (e.g., one of the *strongly contiguous* r.e. degrees introduced by Downey [11]; such a degree must contain a w-trivial structure by Proposition 4.6.1 below), then it would be absurd for C and $\mathcal{A}$ to share a Turing degree.

4.4 Proof of Theorem 4.3.6

Proof of Part (1).

We are to show that if $\mathcal{A}$ is not w-trivial there is an isomorphic copy $\mathcal{A}^*$ of $\mathcal{A}$ such that $\mathcal{A}^* \leq_T \mathcal{A}$ and $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above $\deg_{wtt}(\mathcal{A}^*)$. We do this in two steps. First, in Lemma 4.4.1, we give a condition on $\mathcal{A}^*$ which implies that $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above $\deg_{wtt}(\mathcal{A}^*)$. The second step, in Lemma 4.4.2, is to show that a suitable $\mathcal{A}^*$ can be built computably in $\mathcal{A}$.

Lemma 4.4.1. *Suppose $\mathcal{A}^*$ is a structure and there is a total computable function g such that, for every $m \in \omega$, there exists a permutation fixing $[0,m) \cup [m+g(m), \infty)$ which is not an automorphism of $\mathcal{A}^* \restriction^{g(m)}_{m+g(m)}$. Then $\mathsf{spec}_{wtt}\mathcal{A}^*$ contains the cone above* $\deg_{wtt}(\mathcal{A}^*)$.

Proof. Fix any $C \geq_{wtt} \mathcal{A}^*$, and perform the construction for Theorem 4.2.2 with $\mathcal{A}^*$ in place of $\mathcal{A}$ to get a copy $\mathcal{B} \cong \mathcal{A}^*$. We claim that the construction gives $\mathcal{B} \equiv_{wtt} C$. To see $\mathcal{B} \leq_{wtt} C$, notice that, at each stage s, we have $k \leq g(m_s)$, and so π_{s+1} and m_{s+1} can be computed using queries only to $C \restriction s+1$ and to $\mathcal{A} \restriction^{g(m_s)}_{m_s+g(m_s)}$.

To see $C \leq_{wtt} \mathcal{B}$, define a computable function h by $h(0) = m_0$, $h(s+1) = g(h(s))+1$. Then $m_s \leq h(s)$ for all s. We can therefore recover $C(s)$ from $\mathcal{B}$ using only queries to $\mathcal{B} \restriction^{h(s)}_{h(s+1)}$. $\square$

It is possible for a structure $\mathcal{A}$ to have some isomorphic copies $\mathcal{A}^*$ that satisfy the conditions of the above lemma and other isomorphic copies that do not. Our second lemma connects the existence of a suitable $\mathcal{A}^*$ with the isomorphism-invariant property of not being w-trivial:

Lemma 4.4.2. *If $\mathcal{A}$ is not w-trivial, then there is an isomorphic copy $\mathcal{A}^* \cong \mathcal{A}$ and a function g meeting the hypotheses of Lemma 4.4.1.*

Proof. Using the fact that $\mathcal{A}$ is not w-trivial, fix a computable, increasing function f such that no finite set S witnesses the triviality of $\mathcal{A} \restriction_\omega^{f(|S|)}$. We use f to define a permutation π giving the desired structure $\mathcal{A}^*$ by $\pi : \mathcal{A} \cong \mathcal{A}^*$. This π is constructed as the pointwise limit of a sequence $(\pi_s)_s$ of permutations.

We also define a computable, nondecreasing sequence of restraints $(m_s)_s$ by $m_0 = 0$, $m_{s+1} = m_s + f(m_s) + 1$. These m_s act as restraints in the construction of π_s.

Construction. We define the sequence $(\pi_s)_s$ by stages, beginning with $\pi_0 = id_\omega$. Suppose we have already defined π_s and wish to define π_{s+1}. Let $\mathcal{A}_s^*$ be the unique structure such that $\pi_s : \mathcal{A} \cong \mathcal{A}_s^*$. By choice of f, there is a permutation ρ_s fixing $[0, m_s)$ which is not an automorphism of $\mathcal{A}_s^* \restriction_\omega^{f(m_s)}$. Recall our assumption from §4.1 that the arity $\mathsf{ar}(R_k)$ of a relation R_k does not exceed $k/2$. Hence we may assume that there is a set $T \subseteq [m_s, \infty)$ of size $|T| \leq f(m_s)$ such that ρ_s fixes the complement $\omega \setminus T$ pointwise. Let τ_s be a permutation fixing $[0, m_s)$ and mapping T into the interval $[m_s, m_s + f(m_s))$.

Take the least $x \in \omega$ such that $\tau_s \circ \pi_s(x) \geq m_{s+1} - 1$, and write $y_s = \tau_s \circ \pi_s(x)$. Let σ_s be the permutation transposing y_s and $m_{s+1} - 1$, and fixing all other numbers, and define

$$\pi_{s+1} = \sigma_s \circ \tau_s \circ \pi_s.$$

This completes the construction.

Verification. Let π be the pointwise limit of the $(\pi_s)_s$. Then π is an injective partial function from $\omega \to \omega$; we claim that π is a permutation. At each stage s, the interval $[0, m_s)$ is in the image of π_s, and for all $t \geq s$ we have $\pi_t^{-1} \restriction m_s = \pi_s^{-1} \restriction m_s$, so π is surjective. The addition of σ_s in the construction ensures that π is total.

Now let $\mathcal{A}^*$ be the unique structure such that $\pi : \mathcal{A} \cong \mathcal{A}^*$, and for each s, let $g(s) = m_{s+1}$. Given any $s \in \omega$ we may define a permutation ψ_s by

$$\psi_s = (\sigma_s \circ \tau_s) \circ \rho_s \circ (\sigma_s \circ \tau_s)^{-1}.$$

Then ψ_s is not an automorphism of $\mathcal{A}^* \upharpoonright_{s+g(s)}^{g(s)}$ and fixes $[0, m_s) \cup [m_s + f(m_s), \infty)$ pointwise, and hence fixes the smaller set $[0, s) \cup [s+g(s), \infty)$ pointwise as well. $\square$

This completes the proof of part (1).

Proof of Part (2).

Given a w-trivial structure $\mathcal{A}$, we wish to construct a nonempty $\Pi_1^{0,\mathcal{A}}$ class such that no member of P is wtt-below an isomorphic copy of $\mathcal{A}$. Before providing the proof in full detail, we give a rough plan of how P will be made.

The class P will be defined through a sequence of restraints of the form '$X \in P \Rightarrow X(w) \neq y$,' with $w \in \omega$ and $y \in \{0, 1\}$. The set of restraints will be computably enumerable in $\mathcal{A}$, so P will indeed be a $\Pi_1^{0,\mathcal{A}}$ class. As well, each natural number w will be used in at most one of these constraints, so P will be nonempty. Each restraint will be the result of a diagonalisation against the eventuality $\hat{\Phi}_e^{\mathcal{B}}(w) = X(w)$, for some $w \in \omega$, some wtt-functional $\hat{\Phi}_e$, and some possible isomorphic copy $\mathcal{B}$ of $\mathcal{A}$.

The challenge will be to diagonalise against all $\hat{\Phi}_e, \mathcal{B}$ with only a countable supply of $w \in \omega$. We must play the w-triviality of $\mathcal{A}$ against the computable bound φ_e used in $\hat{\Phi}_e$. In fact, for a fixed $\hat{\Phi}_e$, there is a strategy to diagonalise against $\hat{\Phi}_e^{\mathcal{B}} = X$ for *all* $\mathcal{B} \cong \mathcal{A}$ while using only finitely many w. First we exhibit the basic strategy, for a single $\hat{\Phi}_e$, by proving a weaker result.

Proposition 4.4.3. *If $\mathcal{A}$ is w-trivial and $\hat{\Phi}_e$ is a wtt-functional, then there is a nonempty class $P \subseteq 2^\omega$ such that, if $X \in P$, then $X \neq \hat{\Phi}_e^{\mathcal{B}}$ for any isomorphic* *copy* $\mathcal{B} \cong \mathcal{A}$.

Proof. If φ_e is not total, then $\hat{\Phi}_e^{\mathcal{B}}$ is not total, so any nonempty P will do. Assume, then, that φ_e is total. Recall our assumption in Definition 4.3.1 that φ_e is strictly increasing. We build P as the class of all elements of 2^ω satisfying a finite set of constraints of the form: '$X \in P \Rightarrow X(w) \neq y$'.

We consider all permutations π and structures $\mathcal{B}$ such that $\pi : \mathcal{A} \cong \mathcal{B}$. If g is any total computable function, then there is a finite set $S \subseteq \omega$, say of cardinality $n = |S|$, such that $\pi \restriction S$ uniquely determines the reduct $\mathcal{B} \restriction_\omega^{g(n)}$. What's more, for any $N \in \omega$, the further restriction $\mathcal{B} \restriction_N^{g(n)}$ can—as we allow π and $\mathcal{B}$ to vary—take no more than $(N+1)^n$ different values: one for each partial function from $S \to N$.

Now suppose that $g(n)$ is large enough to admit a sequence

$$N_0 < N_1 < \cdots < N_n < N_{n+1} \leq g(n)$$

such that, for each $i \leq n$, we have $N_{i+1} \geq \varphi_e(N_i + (N_i+1)^n)$. Consider the intervals $[N_i, N_{i+1})$, for $i \leq n$. Since these intervals are pairwise disjoint, there are $n+1$ of them, and the set S has only n elements, for any particular choice of π and $\mathcal{B}$, the Pigeonhole Principle gives an $i_0 \leq n$ such that π maps no element of S into $[N_{i_0}, N_{i_0+1})$. Then the restricted diagram $\mathcal{B} \restriction_{N_{i_0+1}}^{g(n)}$ is uniquely determined by its further restriction $\mathcal{B} \restriction_{N_{i_0}}^{g(n)}$, and so can—as we allow π and $\mathcal{B}$ to vary, preserving $\pi(S) \cap [N_{i_0}, N_{i_0+1}) = \emptyset$—take no more than $(N_{i_0}+1)^n$ possible values. Enumerate these possible diagrams $D_0, D_1, \ldots, D_{\ell-1}$, with $\ell \leq (N_{i_0}+1)^n$.

Suppose that $\pi, \mathcal{B}$ are such that $\pi(S) \cap [N_{i_0}, N_{i_0+1}) = \emptyset$, say with $\mathcal{B} \restriction_{N_{i_0+1}}^{g(n)} = D_j$, and note that

$$N_{i_0+1} > \varphi_e(N_{i_0} + (N_{i_0}+1)^n) \geq \varphi_e(N_{i_0} + \ell).$$

We can ensure that $X \in P \Rightarrow X \neq \hat{\Phi}_e^{\mathcal{B}}$ by waiting for $\hat{\Phi}_e^{D_j}(N_{i_0} + j)$ to converge, and then adding the constraint: '$X \in P \Rightarrow X(N_{i_0}+j) \neq \hat{\Phi}_e^{\mathcal{B}}(N_{i_0}+j)$'.

It therefore suffices to produce a computable g, a natural number n, and a sequence $N_0 < \cdots < N_{n+1} \leq g(n)$ behaving as above. Define a 2-ary computable function h by $h(x, 0) = x$, $h(x, y+1) = \varphi_e(h(x, y) + (h(x, y) + 1)^x)$, and let $g(x) = h(x, x+1)$. Then g is a total computable function, giving a suitable n through w-triviality. We get $N_0, \ldots, N_{n+1}$ by setting $N_i = h(n, i)$ for each $i \leq n+1$. $\square$

We can get a quick and interesting, though weak, result by iterating the above construction in a recklessly noneffective way:

Proposition 4.4.4. *If $\mathcal{A}$ is w-trivial, then there is a set $X \in 2^\omega$ such that $X \not\leq_{wtt} \mathcal{B}$ for any isomorphic copy $\mathcal{B} \cong \mathcal{A}$.*

Proof. The construction from Proposition 4.4.3 uses only finitely many witnesses w to diagonalise—namely, each w is taken from the interval $[N_0, N_{n+1})$. We can therefore perform the construction for each $\hat{\Phi}_e$, $e = 0, 1, \ldots$ in turn, either doing nothing (if φ_e is not total) or running the procedure for Proposition 4.4.3 with the additional stipulation that N_0 be larger than any number thus far considered. $\square$

Note that this is already gives a fairly effective proof of Theorem 4.1.4. The full proof of Theorem 4.3.6, of course, will do better still. We now press on with Part (2)

Idea. The idea is to use the construction from Proposition 4.4.3 as the basic module for meeting the requirement:

$$\mathcal{R}_e : e \in \omega, X \in P, \mathcal{B} \cong \mathcal{A} \implies X \neq \hat{\Phi}_e^{\mathcal{B}}.$$

The main obstacle is that the construction we have given is not uniform with respect to e: it treats a total φ_e differently from a nontotal φ_e, and, in the total case, it assumes knowledge of a suitable finite set S. To fix this, we will treat all

φ_e as if they *might* be total, create an effective list $g_{\langle e,n,x\rangle}$ of uniformly computable functions to use in place of g, and, for each such $g_{\langle e,n,x\rangle}$, make a certain finite number of *guesses* as to what a suitable S might be. For each such S, we then diagonalise as in the basic module.

Each $g_{\langle e,n,x\rangle}$ will come equipped with a guess—namely, n—for the cardinality of an S witnessing the triviality of $\mathcal{A}\restriction_{\omega}^{g_{\langle e,n,x\rangle}(x)}$. Although, as has already been mentioned, the number of guesses we need for S is finite, it far exceeds the bound $g_{\langle e,n,x\rangle}(x)$. This is a source of tension. We overcome this by defining a much faster-growing computable function $f_{\langle e,n\rangle}$ and make the wilder guess that S witnesses the triviality of $\mathcal{A}\restriction_{\omega}^{f_{\langle e,n\rangle}(x)}$. Then we use w-triviality to argue that, for some x and n, there is indeed a suitable S of size n, and the bound $f_{\langle e,n\rangle}(x)$ is large enough to diagonalise for each guess at S.

Before giving the construction in full, we state and prove some helpful combinatorial lemmas.

Definition 4.4.5. We are given a structure $\mathcal{A}$. Define the *growth function* G as a two-place function taking as arguments $M, N \in \omega \cup \{\omega\}$, and yielding the value

$$G_N^M = (\mu n \in \omega)[\exists S \subseteq N \text{ of size } n \text{ s.t. } S \text{ witnesses the triviality of } \mathcal{A}\restriction_N^M],$$

or $G_N^M = \omega$ if there is no such n.

Here are a few easy and useful properties of the growth function.

Facts. (i) The one-place function $M \mapsto G_\omega^M$ is an automorphism invariant of $\mathcal{A}$.

(ii) When $M, N \in \omega$ are finite, G_N^M is computable effectively in $\mathcal{A}$ as a function of $\langle M, N\rangle$.

(iii) G is monotonic in the sense that, if we have $M, M^*, N, N^* \in \omega \cup \{\omega\}$, then $M \leq M^*$ and $N \leq N^*$ implies $G_N^M \leq G_{N^*}^{M^*}$.

(iv) For each M, $\lim_{s\to\omega} G^M_s = G^M_\omega$.

(v) $\mathcal{A}$ is w-trivial if and only if $(\forall M \in \omega)(\forall N \in \omega \cup \{\omega\})[G^M_N$ is finite] and for all total computable f there is an n such that $G^{f(n)}_\omega \leq n$.

(vi) If $\mathcal{A}$ is w-trivial and $F_0 \leq F_1 \leq \cdots$ is a pointwise-increasing sequence of total uniformly computable functions, then there exist natural numbers n, y such that $G^{F_n(y)}_\omega = n = G^{F_{n+1}(y)}_\omega$.

Proof. (i) Immediate.

(ii) Use brute force: for every subset $S \subseteq N$, check whether S witnesses the triviality of $\mathcal{A}\restriction^M_N$.

(iii) If S witnesses the triviality of $\mathcal{A}\restriction^{M^*}_{N^*}$, then S also witnesses the triviality of $\mathcal{A}\restriction^M_N$.

(iv) Immediate.

(v) Immediate from the definition of w-trivial.

(vi) Define a total computable function ψ by $\psi(x) = F_{x+1}(x)$, and use Fact (v) to get a y such that $G^{\psi(y)}_\omega \leq y$. By Fact (iii), we have $0 \leq G^{F_0(y)}_\omega \leq \cdots \leq G^{F_{y+1}(y)}_\omega \leq n$. The result now follows from the following pigeonhole-type fact: If $\sigma : y + 2 \to y$ is an increasing sequence, then there is an n such that $\sigma(n) = n = \sigma(n + 1)$.

□

We have mentioned that, when guessing at suitable sets S to use for the diagonalisation strategy, we need only finitely many guesses. The following result makes this precise.

Lemma 4.4.6. *Suppose that $M \in \omega$, that $G^M_\omega = n$, and that $t \in \omega$ is large enough that $G^M_t = n$. Then there is a set $S \subseteq t$ of cardinality n witnessing the triviality of*

$\mathcal{A}\restriction^M_\omega$, and furthermore we can identify from $\mathcal{A}\restriction^M_t$ a list of sets $(S_0, S_1, \ldots, S_{M^n-1})$, such that $S = S_j$ for some $j < \ell$.

Proof. Pick any $S \subseteq \omega$ of cardinality n which witnesses the triviality of $\mathcal{A}\restriction^M_\omega$. Then $S \cap t$ must witness the triviality of $\mathcal{A}\restriction^M_t$. By our assumption that $G^M_\omega = n$, we must have $|S \cap t| \geq n$. Since $|S| = n$, this implies that $S \subseteq t$.

We may naturally associate with each $j < M^n$ a sequence $\tau_j : n \to M$. We build a guess S_j by a sequence $\emptyset = S_j^{(0)} \subseteq \ldots \subseteq S_j^{(n)} = S_j$, where each $S_j^{(i)}$ has cardinality i. Suppose that we have already chosen $S_j^{(i)}$, and $i < n$. Since $|S_j^{(i)}| = i < n = G^M_t$, this $|S_j^{(i)}|$ does not witness the triviality of $\mathcal{A}\restriction^M_t$. In some fixed computable enumeration, find the first permutation ρ fixing $S_j^{(i)} \cup [t, \infty)$ which is not an automorphism of $\mathcal{A}\restriction^M_t$. Next, find the lexicographically-least sequence $\langle k, x_0, \ldots, x_{\mathsf{ar}(R_k)-1}\rangle$ for which it is *not* the case that

$$R_k^{\mathcal{A}}(x_0, \ldots, x_{\mathsf{ar}(R_k)-1}) \text{ holds if and only if } R_k^{\mathcal{A}}(\rho(x_0), \ldots, \rho(x_{\mathsf{ar}(R_k)-1})) \text{ holds.}$$

Clearly, S must contain at least one element of the set

$$U = \{x_0, \ldots, x_{\mathsf{ar}(R_k)-1}, \rho(x_0), \ldots, \rho(x_{\mathsf{ar}(R_k)-1})\} \setminus S_j^{(i)}.$$

Recalling our assumption from §4.1 that $\mathsf{ar}(R_k) \leq k/2$, this U has size at most $k \leq n$. We extend $S_j^{(i)}$ to $S_j^{(i+1)}$ by adding the $\tau_j(i)$-th smallest element of U (if $\tau_j(i) \geq |U|$, we just add the largest element of U).

We can see by induction that, for every i, there is a j such that $S_j^{(i)} \subseteq S$. In particular, there is a j such that $S_j = S$. □

Strategy. Our strategy uses a certain class of partial functions $g_{\langle e,n,x\rangle}$. We show how to use $g_{\langle e,n,x\rangle}$ before defining it explicitly; for the moment, suffice it to say that $g_{\langle e,n,x\rangle}$ is uniformly computable, and that, whenever φ_e is total, $g_{\langle e,n,x\rangle}$ is total, and for all x and y, there is enough space in the interval $[x, g_{\langle e,n,x\rangle}(y))$ to diagonalise against a single S of size n witnessing the triviality of $\mathcal{A}\restriction_\omega^{g_{\langle e,n,x\rangle}(y)}$. From $g_{\langle e,n,x\rangle}$ we define a second class of functions:

$$f_{\langle e,n\rangle}(x) = \underbrace{g_{\langle e,n,x\rangle} \circ \cdots \circ g_{\langle e,n,x\rangle}}_{n^x \text{ times}}(0).$$

Then $f_{\langle e,n\rangle}$ is uniformly computable and is total whenever φ_e is total, and there is enough space in the interval $[x, f_{\langle e,n\rangle}(x))$ to diagonalise against n^x-many different sets S of size n. Here are the essential steps we use to construct P. Note that we dovetail at step (1). In the first pass, we have $s = 0$.

(1) Start with a 3-tuple $s = \langle e, n, x\rangle$. The number e identifies the requirement $\mathcal{R}_e$ that we are trying to fulfil. The number n represents a guess at the size of a suitable set S against which to diagonalise. The number x is a parameter that ranges over ω.

(2) Wait for a stage t at which $f_{\langle e,n\rangle,t}(x)\downarrow$ and such that $G_t^{f_{\langle e,n\rangle}(x)} = n$. While we are waiting, return to step (1), this time using $s+1$ as the 3-tuple.

(3) Assume—possibly incorrectly—that $G_\omega^{f_{\langle e,n\rangle}(x)} = n = G_\omega^x$. Use the method of Lemma 4.4.6 to make a sequence $(S_j)_{j<x^n}$ of guesses at an S of size n witnessing the triviality of $\mathcal{A}\restriction_\omega^{f_{\langle e,n\rangle}(x)}$.

(4) For each $j < x^n$, use the space in the interval

$$[\underbrace{g_{\langle e,n,x\rangle} \circ \cdots \circ g_{\langle e,n,x\rangle}}_{j \text{ times}}(0), \underbrace{g_{\langle e,n,x\rangle} \circ \cdots \circ g_{\langle e,n,x\rangle}}_{j+1 \text{ times}}(0))$$

to diagonalise for S_j, adding restraints to P by the method of Proposition 4.4.3. If our assumption at step (3) was correct, then this will satisfy the requirement $\mathcal{R}_e$.

Definition of $g_{\langle e,x,n\rangle}$ and allocation of space for diagonalisation.

Define a sequence $(M_k)_k$ of natural numbers recursively by $M_0 = 0$ and $M_{k+1} = M_k + (M_k+1)^k$. The intervals $[M_k, M_{k+1})$ form a partition of ω. For any total

φ_e and any S of size $|S| \leq k$, we could use the interval $[M_k, \varphi_e(M_{k+1}))$ as one of the $[N_i, N_{i+1})$ from the construction in Proposition 4.4.3, and diagonalise for the case $\pi(S) \cap [M_k, \varphi_e(M_{k+1})) = \emptyset$ by placing restraints on $X \cap [M_k, M_{k+1})$ for $X \in P$. To each 3-tuple $\langle e, n, x\rangle$ we assign a sequence of such intervals to use to meet requirement $\mathcal{R}_e$. We make this allocation methodical by defining a uniformly computable function $h_{\langle e,n,x\rangle}$:

$$h_{\langle e,n,x\rangle}(0) = M_{\langle e,n,x,i\rangle}, \quad \text{where } i \text{ is least such that } n \leq \langle e,n,x,i\rangle$$

$$h_{\langle e,n,x\rangle}(y+1) = M_{\langle e,n,x,i\rangle}, \ \text{ where } i \text{ is least such}$$

$$\text{that } \varphi_e(h_{\langle e,n,x\rangle}(y) + (h_{\langle e,n,x\rangle}(y)+1)^n) \leq M_{\langle e,n,x,i\rangle}.$$

The intervals allocated to $\langle e, n, x\rangle$ are those of the form $[M_k, M_{k+1})$ such that $M_k = h_{\langle e,n,x\rangle}(y)$ for some y. Notice that $h_{\langle e,n,x\rangle}$ is total whenever φ_e is total. From here we can define the promised $g_{\langle e,n,x\rangle}$:

$$g_{\langle e,n,x\rangle}(x) = \underbrace{h_{\langle e,n,x\rangle} \circ \cdots \circ h_{\langle e,n,x\rangle}}_{n+1 \text{ times}}(x).$$

Verification. It remains to check that, for every e such that φ_e is total, there is a pair n, x such that $G_\omega^{f_{\langle e,n\rangle}(x)} = n = G_\omega^x$. Fix any e such that φ_e is total, and define a pointwise-increasing sequence of total uniformly computable functions $(F_n)_n$ recursively by $F_0 = id$ and $F_{n+1} = f_{\langle e,n\rangle} \circ F_n$. We can apply Fact (vi) to get a pair n, y such that $G_\omega^{F_{n+1}(y)} = n = G_\omega^{F_n(y)}$. Letting $x = F_n(y)$, this expression becomes $G_\omega^{f_{\langle e,n\rangle}(x)} = n = G_\omega^x$. Hence our strategy, when beginning with the triple $\langle e, n, x\rangle$, succeeds in satisfying $\mathcal{R}_e$.

This completes the proof of Theorem 4.3.6. $\square$

4.5 Structures with finite signature

In this section, we examine the special case of a structure $\mathcal{A}$ with finite signature $(R_0^{\mathcal{A}}, \ldots, R_{n-1}^{\mathcal{A}})$. As noted above, such an $\mathcal{A}$ is w-trivial if and only if $\mathcal{A}$ is trivial if and only if $\mathsf{spec}_T\mathcal{A} = \{\mathbf{0}\}$; this, in turn, happens if and only if $\mathsf{spec}_{wtt}\mathcal{A} = \{\mathbf{0}\}$. We may use Proposition 4.3.2 together with Theorem 4.3.6 to obtain a sharpened dichotomy in the finite-signature case:

Corollary 4.5.1. *Let $\mathcal{A}$ be a structure with finite signature. Either $\mathsf{spec}_{wtt}\mathcal{A}$ contains the cone above some degree $\mathbf{a}$, or $\mathsf{spec}_{wtt}\mathcal{A} = \{\mathbf{0}\}$.*

Therefore, in restricting our structures to those with finite signature, we also restrict the possible wtt degree spectra. We shall see in Proposition 4.6.2 below that, for a structure with infinite signature, the wtt-degree spectrum may be contained within a single cone $\mathcal{D}_{wtt}(\geq \mathbf{a})$ with $\mathbf{a} > \mathbf{0}$. The following proposition shows that such a wtt degree spectrum is impossible for a structure with finite signature.

Proposition 4.5.2. *If $\mathcal{A}$ has finite signature, then $\mathsf{spec}_{wtt}\mathcal{A}$ is not contained in any cone of the form $\mathcal{D}_{wtt}(\geq \mathbf{e})$ with $\mathbf{e} > \mathbf{0}$.*

Our proof uses the following definition and lemma from basic model theory.

Definition 4.5.3. Let $\mathcal{A}$ be a structure, let F be a finite set of elements of $\mathcal{A}$, and let $I = (a_0, a_1, \ldots)$ be an infinite sequence of natural numbers without repetition. We say that I is a sequence of *quantifier-free order indiscernibles over F* if, for every pair of increasing sequences $(i_0 < \ldots < i_{n-1})$ and $(j_0 < \ldots < j_{n-1})$, the tuples $(a_{i_0}, \ldots, a_{i_{n-1}})$ and $(a_{j_0}, \ldots, a_{j_{n-1}})$ satisfy the same quantifier-free formulas with parameters from F.

Lemma 4.5.4. *Let $\mathcal{A} = (\omega, R_0^{\mathcal{A}}, \ldots, R_{n-1}^{\mathcal{A}})$ be a structure with finite signature, and let m be a natural number.*

(i) There is an infinite sequence I of quantifier-free order indiscernibles over $\{0, \ldots, m-1\}$.

(ii) There exists an infinite computable structure $\mathcal{C} = (\omega, R_0^{\mathcal{C}}, \ldots, R_{n-1}^{\mathcal{C}})$ and an increasing injection $\rho : \omega \to \omega$ such that $\rho \upharpoonright m = id_m$ and ρ embeds $\mathcal{C}$ into $\mathcal{A}$.

Proof. Part (i) is an easy consequence of Ramsey's Theorem; see, for example, Shelah [60, Ch.1 §2 Theorem 2.4(1)]. We deduce part (ii) from part (i) as follows. Let $\mathcal{A}$ and m be as in the statement of the Lemma, and let $I = (a_0, a_1, \ldots)$ be the sequence given by part (i). Passing to a subsequence if necessary, we may assume I is increasing. Define $\rho : \omega \to \omega$ by $\rho(i) = i$ if $i < m$, and $\rho(j + m) = a_j$ for all j. Let $\mathcal{C}$ be the unique structure such that ρ is an embedding of $\mathcal{C}$ into $\mathcal{A}$. Then $\mathcal{C}$ is computable. $\square$

We use this Lemma to prove Proposition 4.5.2 by a diagonalisation argument:

Proof of Proposition 4.5.2. Fix a structure $\mathcal{A} = (\omega, R_0^{\mathcal{A}}, \cdots, R_{n-1}^{\mathcal{A}})$ and a set E of wtt degree $\mathbf{e} > \mathbf{0}$. We exhibit a permutation π such that, if $\mathcal{B}$ is the unique structure such that $\pi : \mathcal{B} \cong \mathcal{A}$, then $E \not\leq_{wtt} \mathcal{B}$. We build this π as the pointwise limit of a sequence $(\pi_e)_e$ of permutations, and alongside these we build a sequence $(m_e)_e$ of natural numbers to act as restraints.

Start with $\pi_0 = id_\omega$ and $m_0 = 0$.

Suppose that π_e and m_e have been defined. We define π_{e+1} and m_{e+1} as follows. Begin by letting $\mathcal{B}_e$ be the unique structure such that $\pi_e : \mathcal{B}_e \cong \mathcal{A}$. Apply Lemma 4.5.4(ii) to the structure $\mathcal{B}_e$ and the number m_e, and take the resulting structure $\mathcal{C}_e$ and embedding ρ_e. Because $\mathcal{C}_e$ is computable and E is not, there is an $x_e \in \omega$ such that either $\hat{\Phi}_e^{\mathcal{C}_e}(x_e) \uparrow$ or $\hat{\Phi}_e^{\mathcal{C}_e}(x_e) \downarrow \neq E(x_e)$. If $\varphi_e(x_e) \uparrow$, let $m_{e+1} = \max(m_e, x_e) + 1$; otherwise, let $m_{e+1} = \max(m_e, x_e, \rho_e(\varphi_e(x_e))) + 1$. Choose a permutation $\tau_e : \omega \to \omega$ such that $\tau_e \upharpoonright m_{e+1} = \rho_e \upharpoonright m_{e+1}$, and τ_e fixes $[0, m_e) \cup [m_{e+1}, \infty)$. Define $\pi_{e+1} = \tau_e \circ \pi_e$. Let π be the pointwise limit of $(\pi_e)_e$. This completes the construction.

Verification. The definition of π_{e+1} can be rewritten as $\pi_{e+1} = \tau_e \circ \tau_{e-1} \circ \cdots \circ \tau_0 \circ id_\omega$. Since each τ_e acts nontrivially only on the interval $[m_e, m_{e+1})$, and these intervals form a partition of ω, the limit π is a permutation. Let $\mathcal{B}$ be the unique structure such that $\pi : \mathcal{B} \cong \mathcal{A}$; we claim that $\mathbf{e} \not\leq_{wtt} \mathcal{B}$. Indeed, for each e, either $\varphi_e(x_e) \uparrow$, in which case $\hat{\Phi}_e^{\mathcal{B}}(x_e) \uparrow$ by definition; or, for each $i \geq e+1$, we have $\pi_i \upharpoonright m_{e+1} = \pi_{e+1} \upharpoonright m_{e+1}$, giving $\pi_i \upharpoonright \varphi_e(x_e) = \rho_e \upharpoonright \varphi_e(x_e)$, so that $\hat{\Phi}_e^{\mathcal{B}}(x_e) = \hat{\Phi}_e^{\mathcal{C}_e}(x_e) \neq E(x_e)$. □

Theorem 4.1.7 follows immediately.

Proof of Theorem 4.1.7. Dovetail the construction above, with $e = 0, 1, 2 \ldots$. □

Finally, we mention some cases where the wtt degree spectrum is provably upward closed. We gave a brief argument in §4.3 that, if $\mathcal{A}$ is a linear order, then the proof of Theorem 4.2.2 actually guarantees upward closure for $\mathsf{spec}_{wtt}\mathcal{A}$. This argument can now be formalised using Lemma 4.4.1 and applied to other examples.

Proposition 4.5.5. *(i) If $\mathcal{A} = (\omega, \leq^{\mathcal{A}})$ is a linear order, then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed.*

(ii) If $\mathcal{A} = (\omega, E^{\mathcal{A}})$ is a structure where $E^{\mathcal{A}}$ is an equivalence relation having more than one infinite class, then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed.

(iii) If $\mathcal{A} = (\omega, E^{\mathcal{A}})$ is a structure where $E^{\mathcal{A}}$ is an equivalence relation having infinitely many nonsingleton classes, then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed.

Proof. (i) Apply Lemma 4.4.1 to $\mathcal{A}$, with $g(m) = m + 2$.

(ii) Let U_1 and U_2 be distinct infinite equivalence classes. Take the isomorphic copy $\mathcal{A}^*$ specified by:

$$E^{\mathcal{A}^*}(3x, 3y) \text{ holds} \iff E^{\mathcal{A}}(x, y) \text{ holds};$$

$$E^{\mathcal{A}^*}(3x+1, z) \text{ holds} \iff z = 3y+1, \text{ or } (z = 3y \text{ and } y \in U_1);$$

$$E^{\mathcal{A}^*}(3x+2, z) \text{ holds} \iff z = 3y+2, \text{ or } (z = 3y \text{ and } y \in U_2).$$

Then $\mathcal{A}^*$ is isomorphic to $\mathcal{A}$, and $\mathcal{A}^* \equiv_{wtt} \mathcal{A}$. Apply Lemma 4.4.1 to $\mathcal{A}^*$, with $g(m) = m+3$.

(iii) Build a permutation π by the following recursive procedure:

$$\pi(0) = 0$$

$$\pi(2x+1) = (\mu y)[y \text{ not in the image of } \pi \restriction 2x+1]$$

$$\pi(2x+2) = (\mu z)[z \text{ not } E^{\mathcal{A}}\text{-equivalent to any } y \text{ in the image of } \pi \restriction 2x+2.$$

Let $\mathcal{A}^*$ be the inverse image of $\mathcal{A}$ under π, i.e., $\pi : \mathcal{A}^* \cong \mathcal{A}$. Then $\mathcal{A}^* \leq_{wtt} \mathcal{A}$. Apply Lemma 4.4.1 to $\mathcal{A}$, with $g(x) = x+6$.

□

Parts (ii) and (iii) can be combined into a single corollary:

Corollary 4.5.6. *Let $\mathcal{A} = (\omega, E^{\mathcal{A}})$ be an equivalence relation. Then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed if and only if $\mathcal{A}$ is not trivial.*

The constructions for (ii) and (iii) in Proposition 4.5.5 are more typical than that for (i). By and large, Ramsey-type considerations make it difficult to meet the hypothesis of Lemma 4.4.1 without first rearranging a model's elements.

As one last example, we mention a large class of graphs $\mathcal{A}$, each of which has an isomorphic copy $\mathcal{A}^* \leq_{wtt} \mathcal{A}$ to which we can apply Lemma 4.4.1. The proof is omitted.

Proposition 4.5.7. *If $\mathcal{A} = (\omega, E^{\mathcal{A}})$ is a graph, and if*

$$(\forall n)(\exists \text{ distinct } a_0, a_1, a_2, a_3 \geq n)(\forall x < n)[a_0 E^{\mathcal{A}} a_1 \ \wedge\ \neg a_2 E^{\mathcal{A}} a_3 \ \wedge \bigwedge_{0 \leq i \leq 3} \neg a_i E^{\mathcal{A}} x]$$

holds, then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed. In particular, if $\mathcal{A}$ has infinitely many nonsingleton components, then $\mathsf{spec}_{wtt}\mathcal{A}$ is upward closed.

4.6 Some specific examples

This section is devoted to a few elementary constructions each giving a partial answer to the question: *What sets of wtt degrees can form a wtt degree spectrum?*

Recall from Definition 4.3.3 that a set A is *of* $\mathbf{0}$*-dominated degree* if and only if, whenever f is a function such that $\mathsf{graph}(f) \leq_{wtt} A$, this f is dominated by a computable function. We say that a wtt-degree $\mathbf{a}$ is $\mathbf{0}$-dominated if its elements are of $\mathbf{0}$-dominated degree.

Proposition 4.6.1. *A wtt-degree* $\mathbf{a}$ *contains a structure that is w-trivial but not trivial if and only if* $\mathbf{a}$ *is not* $\mathbf{0}$*-dominated.*

Proof. The 'only if' direction is immediate from Theorem 4.3.6 and the observation in Proposition 4.3.4 that, if $\mathcal{A}$ is $\mathbf{0}$-dominated, then $\mathsf{spec}_{wtt}\mathcal{A}$ contains a cone.

For the 'if' direction, suppose that $\mathbf{a}$ is not $\mathbf{0}$-dominated and fix a member $A \in \mathbf{a}$. Let f be a strictly increasing function that is not computably dominated, and such that $\mathsf{graph}(f) \leq_{wtt} A$. We construct a structure $\mathcal{B} = (\omega, R_0^{\mathcal{B}}, R_1^{\mathcal{B}}, \ldots)$, with each $R_k^{\mathcal{B}}$ unary, such that $\mathcal{B}$ is w-trivial, $\mathcal{B}$ is not trivial, and $\mathcal{B} \equiv_{wtt} A$. For each k, define:

$$R_{2k}^{\mathcal{B}} = \begin{cases} \{k\} & \text{if } k \text{ is in the image of } f, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$R_{2k+1}^{\mathcal{B}} = \begin{cases} \omega \text{ if } k \in A \\ \emptyset \text{ if } k \notin A \end{cases}$$

Then $\mathcal{B} \equiv_{wtt} A$. To see that $\mathcal{B}$ is w-trivial, let ψ be any increasing total computable function, and take n such that $\psi(n) < f(n)$. Let $S = \{k : 0 \leq k < n\}$. This S has cardinality n and witnesses the triviality of $\mathcal{B}\restriction_{\omega}^{\psi(n)}$, as desired. $\square$

Our next construction gives a wide class of possible wtt degree spectra, and, as mentioned in §4.5 above, highlights an important difference between the finite-and infinite-signature cases.

Proposition 4.6.2. *For any wtt degree* $\mathbf{a}$*, there is a* $\mathcal{B}$ *such that* $\mathsf{spec}_{wtt}\mathcal{B} = \mathcal{D}_{wtt}(\geq \mathbf{a})$*.*

Proof. If $\mathbf{a} = \mathbf{0}$, then we can use any computable $\mathcal{B}$ which is not trivial. So suppose that $\mathbf{a} > \mathbf{0}$, and fix a member $A \in \mathbf{a}$. Define $\mathcal{B} = (\omega, R_0^{\mathcal{B}}, R_1^{\mathcal{B}}, \ldots)$, with each $R_k^{\mathcal{B}}$ unary, as follows.

$$R_0^{\mathcal{B}} = \{0, 2, 4, 6, \ldots\}$$

$$R_{k+1}^{\mathcal{B}} = \begin{cases} \omega \text{ if } k \in A \\ \emptyset \text{ if } k \notin A \end{cases}$$

Then A is wtt-below any isomorphic copy $\mathcal{C}$ of $\mathcal{B}$, since we can decide whether a given k is in A by checking whether $R_k^{\mathcal{C}}(0)$ holds. On the other hand, if X is a set such that $X \geq_{wtt} A$, then X must be infinite and co-infinite, and so we may construct an isomorphic copy $\mathcal{C}$ of $\mathcal{B}$ such that $\mathcal{C} \equiv_{wtt} X$ as follows:

$$R_0^{\mathcal{C}} = X$$

$$R_{k+1}^{\mathcal{C}} = \begin{cases} \omega \text{ if } k \in A \\ \emptyset \text{ if } k \notin A. \end{cases}$$

□

Our next construction shows that, as a set of reals, every T degree spectrum not consisting of a single degree is equal to a wtt degree spectrum. Hence wtt degree spectra of nontrivial structures are at least as expressive, when considered as subsets of 2^ω, as T degree spectra of nontrivial structures.

Proposition 4.6.3. *If* $\mathcal{A}$ *is a structure which is not trivial, then there is a graph* $H = (\omega, E^H)$ *such that* $\bigcup \mathsf{spec}_{wtt} H = \bigcup \mathsf{spec}_T \mathcal{A}$*.*

Proof. By Theorem 4.1.8, we may fix a graph $G = (\omega, E^G)$ with Turing degree spectrum $\mathsf{spec}_T G = \mathsf{spec}_T \mathcal{A}$. We may assume that G has no isolated points, that is, for all x there exists a y such that $(x, y) \in E^G$. We use G to build a new graph $H = (\omega, E^H)$ with the following properties:

(i) $\mathsf{spec}_T H = \mathsf{spec}_T G$

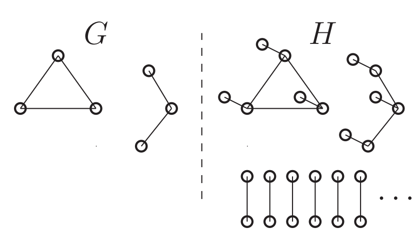
(ii) $\mathsf{spec}_{wtt} H$ is upward closed.

(iii) Given $X \in 2^\omega$ and a copy $K \cong H$, if $X \geq_T K$, then there is another copy $L \cong H$ such that $X \geq_{wtt} L$.

This is then the desired H by the following string of equivalences:

$$\begin{aligned} X \in \bigcup \mathsf{spec}_T \mathcal{A} \quad &\text{iff} \quad X \in \bigcup \mathsf{spec}_T G, \quad \text{by choice of } G \\ &\text{iff} \quad X \in \bigcup \mathsf{spec}_T H, \quad \text{by (i)} \\ &\text{iff} \quad X \geq_T K \text{ for some } K \cong H, \quad \text{since } \mathsf{spec}_T H \text{ is upward closed} \\ &\text{iff} \quad X \geq_{wtt} L \text{ for some } L \cong H, \quad \text{by (iii)} \\ &\text{iff} \quad X \in \bigcup \mathsf{spec}_{wtt} H, \quad \text{by (ii)} \end{aligned}$$

Construction. We transform G into the new graph H by appending exactly one new vertex to each vertex of G, and then adding a countable perfect matching. In pictures, the transformation behaves like this:



We define the edge relation on H by cases, closing under symmetry:

- If $x = 4n$, $y = 4m$, and $(m, n) \in E^G$, then $(x, y) \in E^H$.
- If $x = 4n$ and $y = 4n + 1$, then $(x, y) \in E^H$.
- If $x = 4n + 2$ and $y = 4n + 3$, then $(x, y) \in E^H$.

We claim that this H satisfies conditions (i),(ii),(iii).

Verification of (i). Notice first that $H \equiv_T G$, and second that, if a copy $G_0 \cong G$ is transformed in the same manner as above into a graph H_0, then $H_0 \cong H$. Thus $\mathsf{spec}_T G \subseteq \mathsf{spec}_T H$. For the opposite inclusion, suppose that H_0 is an isomorphic copy of H. Define a set $A \subseteq \omega$ of vertices by:

$$A = \{x \in \omega : (\exists \text{ at least two distinct } y)[(x,y) \in E^{H_0}]\}.$$

Because G has no isolated points, the subgraph induced by H_0 on A is isomorphic to G. Define an injection $\rho : \omega \to \omega$ by

$$\rho(n) = \text{ the } n\text{-th element enumerated into A},$$

and let G_1 be the unique structure such that ρ is an embedding of G_1 into H. Then $G_1 \cong G$ and $G_1 \leq_T H$. We conclude by the upward-closure result of Theorem 4.2.2 that $\mathsf{spec}_T G \subseteq \mathsf{spec}_T H$.

Verification of (ii). For any n, the elements $a_0 = 4n{+}2$, $a_1 = 4n{+}3$, $a_2 = 4n{+}6$, and $a_3 = 4n + 10$ satisfy the statement:

$$(\forall x < n)[a_0 E^H a_1 \quad \wedge \quad \neg a_2 E^H a_3 \quad \wedge \bigwedge_{0 \leq i \leq 3} \neg a_i E^H x].$$

Hence Proposition 4.5.7 implies that $\mathsf{spec}_{wtt} H$ is upward closed.

Verification of (iii). Suppose that K is is an isomorphic copy of H and that $X \geq_T K$, say by the computation $\mathcal{D}K = \Phi_e^X$. We get the required L by the following 'padding' procedure. For each $n \in \omega$, let u_n be least such that $\Phi_e^{X \upharpoonright u_n}$ computes the restricted diagram $K \upharpoonright_n$, i.e., such that Φ_e^X computes $K \upharpoonright_n$ with use u_n. Define a sequence $(v_n)_{n \in \omega}$ by $v_0 = u_0$, $v_{n+1} = v_n + 2u_n + 3$. We define the edge relation on L by the following cases, closing under symmetry:

- If $x = v_m$ and $y = v_n$, then $(x,y) \in E^L$ if and only if $(m,n) \in E^K$.
- If $v_m < x < v_{m+1} - 1$, then $(x, x+1) \in E^L$ if and only if $x - v_m$ is odd.

That is, K is embedded into L by the mapping $m \mapsto v_m$, and the remaining elements of L form an infinite perfect matching. Since K itself contains an infinite perfect matching, L and K are isomorphic. Now we check that $L \leq_{wtt} X$. Given a number x, look at the computation of $\Phi_e^{X\restriction x}$ to find the least m such that $v_m > x$. We can use the computation of $\Phi_e^{X\restriction x}$ to recover both the restricted diagram $K\restriction_m$ and the sequence $(v_0, \ldots, v_{m-1})$. This information is enough to construct the restricted diagram $L\restriction_x$. $\square$

We end with a construction of a wtt degree spectrum that, as a set of reals, does not coincide with any Turing degree spectrum. When combined with Proposition 4.6.3, this establishes the result promised in §4.1.1 that, as a means of specifying a set of reals, the wtt degree spectrum of a nontrivial structure is strictly more expressive than the Turing degree spectrum of a nontrivial structure. As usual, there is some tension between the complexity of the construction and the contrivedness of the object being built. The following class of structures appears to be a good compromise.

Definition 4.6.4. Let $\mathcal{A} = (\omega, \underline{0}^{\mathcal{A}}, S^{\mathcal{A}}, E^{\mathcal{A}})$ be a structure with $\underline{0}^{\mathcal{A}}$ a unary relation, and $S^{\mathcal{A}}, E^{\mathcal{A}}$ binary relations. We say that $\mathcal{A}$ is a *labelled graph* if the reduct $(\omega, E^{\mathcal{A}})$ is a graph and the reduct $(\omega, \underline{0}^{\mathcal{A}}, S^{\mathcal{A}})$ is isomorphic to the natural numbers with zero and successor (with $\underline{0}^{\mathcal{A}}$ and $S^{\mathcal{A}}$ interpreted as a constant and a unary function, respectively). Given an element $n \in \omega$, let $\delta^{\mathcal{A}}(n)$ be the neighbourhood of n in $(\omega, E^{\mathcal{A}})$, i.e.,

$$\delta^{\mathcal{A}}(n) = \{m \in \omega : (m, n) \in E^{\mathcal{A}}\}.$$

For any natural number e, let $\underline{e}^{\mathcal{A}}$ denote the unique e-th element:

$$\underline{e}^{\mathcal{A}} = \underbrace{S^{\mathcal{A}}(S^{\mathcal{A}}(\cdots S^{\mathcal{A}}}_{e \text{ times}}(\underline{0}^{\mathcal{A}}))).$$

Proposition 4.6.5. *There is a labelled graph $\mathcal{A}$ such that $\bigcup \mathsf{spec}_{wtt}\mathcal{A} \neq \bigcup \mathsf{spec}_T\mathcal{A}$.*

Proof. Let $\frown$ be the concatenation operator for strings, and let $\left(\tilde{\Phi}_e\right)_e$ be the enumeration of all wtt reductions given by:

$$\tilde{\Phi}_e^Y(x) = \begin{cases} \Phi_e^Y(x) \text{ if } \mathsf{use}\Phi_e^Y(x) < \varphi_e(x) \\ \uparrow \text{ otherwise.} \end{cases}$$

We build $\mathcal{A}$, together with a set $Z \subseteq \omega$, to satisfy the following requirements:

$\mathcal{P}$: $\mathcal{A} \leq_T Z$.

$\mathcal{N}_e$: If $\mathcal{B}$ is a labelled graph and $\mathcal{B} = \tilde{\Phi}_e^Z$, then $\mathcal{A} \ncong \mathcal{B}$.

The requirement $\mathcal{P}$ ensures that $\deg_T(Z) \in \mathsf{spec}_T\mathcal{A}$, while the requirements $\mathcal{N}_e$ together ensure that $\deg_{wtt}(Z) \notin \mathsf{spec}_{wtt}\mathcal{A}$.

Strategy. We build Z by initial segments $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq Z$. At each stage n, we specify σ_n and $\mathcal{A}\restriction_n$. The reduct $(\omega, \underline{0}^{\mathcal{A}}, S^{\mathcal{A}})$ will be ordered in the most straightforward way, namely, $\underline{e}^{\mathcal{A}} = e$ for all e.

We begin by declaring that each negative requirement $\mathcal{N}_e$ *has not acted.* At a stage of the form $n+1 = \langle e, x\rangle + 1$, if $\mathcal{N}_e$ has not yet acted, we may choose to fix the set $\delta^{\mathcal{A}}(e)$ as either a finite or a cofinite set. The goal is to satisfy $\mathcal{N}_e$ by ensuring, if $\mathcal{B}$ is labelled graph and $\mathcal{B} = \tilde{\Phi}_e^Z$, that:

$$\left|\delta^{\mathcal{A}}(e)\right| \neq \left|\delta^{\mathcal{B}}(\underline{e}^{\mathcal{B}})\right| \text{ or } \left|\omega \setminus \delta^{\mathcal{A}}(e)\right| \neq \left|\omega \setminus \delta^{\mathcal{B}}(\underline{e}^{\mathcal{B}})\right|.$$

After we decide to fix $\delta^{\mathcal{A}}(e)$, we say that $\mathcal{N}_e$ *has acted.* At the end of the stage, we define σ_{n+1} and the restricted diagram $\mathcal{A}\restriction_{n+1}$ based on the decisions made at earlier stages for other neighbourhoods $\delta^{\mathcal{A}}(i)$.

We meet $\mathcal{P}$ by coding the atomic diagram of $\mathcal{A}$ directly into Z. For each n, σ_{n+1} will equal $\sigma_n \frown 0^s \frown 1 \frown 0^r \frown 1$ for some s to be specified below and a number r representing the atomic diagram $\mathcal{A}\restriction_n$ by some fixed computable encoding.

Construction. At stage $n = 0$, we let $\sigma_0 = \emptyset$.

At each stage of the form $n + 1 = \langle e, x\rangle + 1$, we try to fulfil requirement $\mathcal{N}_e$ as outlined above. If $\mathcal{N}_e$ has not yet acted, then use a $\mathbf{0}'$ oracle to extend σ_n, if possible, to a string $\tau = \sigma_n {}^\frown 0^s$ such that $\tilde{\Phi}^\tau_{e,s}$ converges to give a large initial segment of an atomic diagram D, having at least $2n+1$ elements, of a $\mathcal{B}$ as in $\mathcal{N}_e$. If $|\delta^D(\underline{e}^D)| \geq n+1$, then we fulfil the requirement $\mathcal{N}_e$ by declaring that $\delta^{\mathcal{A}}(e)$ shall be a subset of $\{0, \ldots, n-1\}$. Otherwise, the complement has size $|\delta^D(\underline{e}^D)| \geq n+1$, and so we fulfil $\mathcal{N}_e$ by declaring that $\omega \setminus \delta^{\mathcal{A}}(e)$ shall be a subset of $\{0, \ldots, n-1\}$. We then preserve the computation by letting $\sigma_{n+1} = \tau {}^\frown 1 {}^\frown 0^r {}^\frown 1$, with r a number representing $\mathcal{A}\restriction_n$. Declare that $\mathcal{N}_e$ *has acted*.

If $\mathcal{N}_e$ has acted at an earlier stage, or if no suitable τ exists, then $\mathcal{N}_e$ does not act at stage $n+1$, and we instead carry out the following procedure. Let $\mathcal{B}$ be the (possibly partial) atomic diagram given by $\mathcal{B} = \tilde{\Phi}^{\sigma_n {}^\frown 0^\omega}_e$. One of four conditions must hold:

(i) There is a y such that $\tilde{\Phi}^{\sigma_n {}^\frown 0^\omega}_e(y)\uparrow$.

(ii) $\mathcal{B}$ contains more than one element of the form $\underline{e}^{\mathcal{B}}$.

(iii) $\mathcal{B}$ contains no element of the form $\underline{e}^{\mathcal{B}}$.

(iv) The requirement $\mathcal{N}_e$ has already acted at an earlier stage.

If (i), then choose an extension $\tau = \sigma_n {}^\frown 0^s$ long enough that, if ρ is a string extending τ, then $\tilde{\Phi}^\rho_e(y)\uparrow$. If (ii), choose $\tau = \sigma_n {}^\frown 0^s$ long enough that, for some y, the atomic diagram $\tilde{\Phi}^\tau_e\restriction_y$ contains more than one $\underline{e}$. If (iii) or (iv), choose $\tau = \sigma_n$. In any case, let $\sigma_{n+1} = \tau {}^\frown 1 {}^\frown 0^r {}^\frown 1$, with r a number representing $\mathcal{A}\restriction_n$.

Verification. It is easy to see that $\mathcal{P}$ is satisfied: For each n, we can find an initial segment $\sigma {}^\frown 1 {}^\frown 0^r {}^\frown 1$ of Z such that exactly $2n$ entries of σ are 1. Then we can use r to recover the restricted diagram $\mathcal{A}\restriction_n$.

Now we check that $\mathcal{N}_e$ is fulfilled. If, at any stage $n+1$, we declared $\mathcal{N}_e$ *has acted*, then our diagonalisation strategy using $\delta^{\mathcal{A}}(e)$ succeeds. So suppose that $\mathcal{N}_e$

never acts, and suppose, for a contradiction, that the requirement $\mathcal{N}_e$ is not met. Let $\mathcal{B} = \tilde{\Phi}_e^Z$ be as in the statement of $\mathcal{N}_e$. Then there is an $n = \langle e, x \rangle$ and a y such that $\tilde{\Phi}_e^{\sigma_n} \restriction_y$ contains a well-defined e-th element. Let $\mathcal{C} = \tilde{\Phi}_e^{\sigma_n \frown 0^\omega}$. Either $\mathcal{C}$ contains a finite substructure D as in the construction, or $\mathcal{C}$ is not total as a characteristic function, or $\mathcal{C}$ contains more than one element of the form $\underline{e}^{\mathcal{C}}$. Of these three possibilities, the first implies that $\mathcal{N}_e$ acts at stage $n+1$, a contradiction; the second puts us in case (i) of the construction; and the third, in case (ii). But in case (i), our choice of σ_{n+1} implies that $\mathcal{B}$ is also not total as a characteristic function, also a contradiction; and in case (ii), our choice of σ_{n+1} implies that $\mathcal{B}$ has multiple elements of the form $\underline{e}^{\mathcal{B}}$ and hence is not a labelled graph, another contradiction. $\square$

As an aside, we note that a labelled graph $\mathcal{A} = (\omega, \underline{0}^{\mathcal{A}}, S^{\mathcal{A}}, E^{\mathcal{A}})$ can be encoded into a single binary relation R with only a small loss of information. Namely:

$$\begin{aligned}
(n, n) \in R \quad &\Longleftrightarrow \quad n \in \underline{0}^{\mathcal{A}}; \quad \text{and, for all pairs } n \neq m, \\
(n, m) \in R \quad &\Longleftrightarrow \quad (n, m) \in S^{\mathcal{A}} \text{ or } (n, m) \in E^{\mathcal{A}} \text{ and } (m, n) \notin S^{\mathcal{A}}.
\end{aligned}$$

In this encoding, we lose the edges between consecutive elements $(n, S^{\mathcal{A}}(n))$ of the labelled graph.

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