# Computable Structures: Presentations Matter

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**Abstract**

The computability properties of a relation $R$ not included in the language of a computable structure $\mathcal{A}$ can vary from one computable presentation to another. We describe some classic results giving conditions on $\mathcal{A}$ or $R$ that restrict the possible variations in the computable dimension of $\mathcal{A}$ (i.e. the number of isomorphic copies of $\mathcal{A}$ up to computable isomorphism) and the computational complexity of $R$. For example, what conditions guarantee that $\mathcal{A}$ is computably categorical (i.e. of dimension 1) or that $R$ is intrinsically computable (i.e. computable in every presentation). In the absence of such conditions, we discuss the possible computable dimensions of $\mathcal{A}$ and variations (in terms of Turing degree) of $R$ in different presentations (the degree spectrum of $R$). In particular, various classic theorems and more recent ones of the author, B. Khoussainov, D. Hirschfeldt and others about the possible degree spectra of computable relations on computable structures and the connections with computable dimension and categoricity will be discussed both in general model theoretic settings and in restricted classes of structures such as graphs, linear and partial orderings, lattices, Boolean algebras, Abelian and nilpotent groups, rings, integral domains, and real or algebraically closed fields.

## 1 Introduction

The general subject area of this paper is effective model theory. There are (at least) two versions of what is the basic subject matter of model theory. Along with Chang and Keisler [1990] one can take the "logical" point of view that it deals with the connections between formal language and their interpretations, or models. Alternatively, with Hodges [1993] we can say that it is the study of the construction and classification of structures within specified classes of structures. The first view starts with

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theories and so the effective version deals with decidable theories, ones for which we have a computable procedure for deciding the validity of each sentence or its truth in a model under consideration. The second is an algebraic view that starts with structures and so the effective version deals with computable structures in the sense that we can compute the basic relations and operations on the structure. Each approach provides a rich area for investigation. In this paper we consider only certain topics from the second, structural or algebraic point of view. One that considers both is Khoussainov and Shore [1999]. More general introductions can be found in *The Handbook of Recursive Algebra* (Ershov et al. [1998]), especially the articles by Harizanov [1998] and Ershov and Goncharov [1998]. This *Handbook* also contains other useful survey papers on aspects of effective model theory and algebra and an extensive bibliography. The one most closely related to the theme of this paper is Goncharov [1998]. Another interesting survey is Millar [1999] in *The Handbook of Computability Theory* (Griffor [1999]). One book in progress on the subject is Ash and Knight [2000].

As for approaches to effectiveness, one can profitably study the issues of computability in model theory and algebra in terms of a wide range of notions from polynomial-time to Borel. In this paper, we restrict ourselves to the fundamental notion of computability (and relative computability) as determined by Turing machines (with oracles). We begin with the basic definition.

**Definition 1.1** *A structure $\mathcal{A}$ is* computable *if its domain $A$ is computable and the functions and relations of $\mathcal{A}$ are uniformly computable (or, equivalently, the atomic diagram of $\mathcal{A}$, $D(\mathcal{A}, a)_{a \in A}$, is computable). $\mathcal{A}$ is* computably presentable *if $\mathcal{A}$ is isomorphic to a computable structure $\mathcal{B}$ via a map $f : A \to B$ which we call a* computable presentation *of $\mathcal{A}$.*

- From now on, all structures will be computable unless otherwise specified.

The question we want to address in this paper is whether computability properties of structures depend on the choice of presentation and if so, how? From the classical mathematical viewpoint, two presentations $\mathcal{A}$ and $\mathcal{B}$ of a structure are, after all, the same structure, i.e. they are isomorphic, so one might ask how can they differ. On the other hand, it is a commonplace in computer science that the choice of representation of data can be crucial in determining the ease of its use. Thus, from the mathematical point of view, the surprise will be that there are differences at all. From the computer science point of view, the surprise is the extent of the differences. The choice of presentation can not only make operations faster or slower but can change them from computable to noncomputable in drastic ways. A classic and striking example is the notion of dependence in vector spaces (linear dependence) or algebraically closed fields (algebraic independence). While we know how to effectively determine whether elements are dependent or not in each setting in the standard presentations of, for example, the infinite dimensional vector space (or algebaically closed field) over the rationals, $\mathbb{Q}$, this cannot be done in all computable presentations of these well known structures (Metakides and Nerode [1977], [1979]). The source of the problem is that, although $\mathcal{A}$ and $\mathcal{B}$ may be isomorphic, the isomorphism between them may not be computable. Thus a relation (like dependence), even though it is preserved under isomorphisms, is not explicitly part of the language and so may be computable in one computable presentation and not another. The point here is that the underlying notion of equivalence between computable structures that is relevant to issues about preserving computability properties is computable rather than classical isomorphism.

**Definition 1.2** *$\mathcal{A}$ is* computably isomorphic *to $\mathcal{B}$, $\mathcal{A} \cong_c \mathcal{B}$, if there is a computable $f : A \to B$ which is an isomorphism. We also say then that $\mathcal{A}$ and $\mathcal{B}$ are* of the same computable isomorphism type.

**Definition 1.3** *The* (computable) dimension *of a structure $\mathcal{A}$ is the number of its computable isomorphism types. $\mathcal{A}$ is* computably categorical *if its computable dimension is $1$, i.e. every $\mathcal{B}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.*

**Example 1.4** $\langle \mathbb{Q}, \leq \rangle$ is computably categorical: The usual back and forth argument used to show that any two dense linear orderings without endpoints are isomorphic is effective and produces computable isomorphisms between any two computable presentations of $\mathbb{Q}$.

**Example 1.5** $\langle \mathbb{N}, s \rangle$ is computably categorical: Given $\mathcal{B} \cong \mathbb{N}$, one defines the required computable $f : N \to B$ by recursion. First, $f(0)$ is the least element of $B$ and then if $f(n) = b$ we let $f(n+1) = s(b)$.

On the other hand, if one considers the natural numbers with the usual ordering relation but without the successor function, the structure is not computably categorical. Indeed, as the above argument shows, any presentation $\mathcal{A}$ in which the successor function is computable is computably isomorphic to the standard presentation. Thus, there are presentations in which the successor function is not computable

**Proposition 1.6** $\langle \mathbb{N}, \leq \rangle$ *has a computable presentation $\mathcal{A}$ in which the successor function is not computable and so $\langle \mathbb{N}, \leq \rangle$ is not computably categorical.*

**Proof.** We construct $\mathcal{A} = \langle A, \preceq \rangle$ by induction. Without loss of generality, we assume that we have a common enumeration of all the computations of the computable functions $\phi_n$ in which just one computation $\phi_n(x)$, for some $n$ and $x$, converges at each stage $s$. We begin with all the even numbers in $A_0$ in their usual order. At stage $s$ of the construction, we see if $\phi_n$ converges on input $2n$ at stage $s$ and gives output $2n + 2$. If so, we add $2s + 1$ to $A_s$ to get $A_{s+1}$ and put $2s + 1$ between $2n$ and $2n + 2$ in the ordering $\preceq$. If not, $A_{s+1} = A_s$. Of course, $A = \cup A_s$. It is easy to see that $\mathcal{A}$ is computable. (To see if some odd number $2s + 1$ is in $A$ just go to stage $s$ of the construction. Similarly, if $x, y \in A$, we can determine if $x \preceq y$ at the stage by which both have been put into $A$.) It is also immediate that $\mathcal{A} \cong \mathbb{N}$ as we start out with an ordering on $A_0$ isomorphic to $\mathbb{N}$ and add at most one element between any two consecutive elements of the original ordering on $A_0$. On the other hand, the usual diagonal argument now shows that no $\phi_n$ is the successor function on $\mathcal{A}$. (If $\phi_n(2n)$ converges and equals $2n + 1$ then the immediate successor of $2n$ in $\mathcal{A}$ is $2s + 1$ where $s$ is the stage at which $\phi_n(2n)$ converges. In every other case, the immediate successor of $2n$ in $\mathcal{A}$ is $2n + 2$ which is not the value of $\phi_n(2n)$.) $\square$

**Example 1.7** As in Example 1.5 the integers $\mathbb{Z}$ as a group with $+$ or as a ring with $+$ and $\times$ is computably categorical as it is finitely generated. The rationals, $\mathbb{Q}$, are also computably categorical as a field as one can effectively generate them from 0, 1 and the field operations $+$ and $\times$ using a procedure that identifies a fraction $p/q$ as the element $r$ such that $p = q \times r$. A bit of Galois theory can then be used to show that $\bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, is computably categorical. On the other hand, the algebraic closure of $\mathbb{Q}$ extended by infinitely many transcendentals has computable dimension $\omega$. The argument need here is more complicated than for $\langle N, \leq \rangle$ but follows from the results of Metakides and Nerode [1979] mentioned above as well as more general theorems that we will discuss below.

## 2 Categoricity and Intrinsic Computability

Since we have intimated that the source of variation in computability properties among different presentations is the fact that the isomorphism between them might not be computable, it is natural to conjecture that computable categoricity should obviate this problem and any relation or operation computable in one presentation of a computably categorical structure should be computable in all presentations. This conjecture is false as stated but is true for all the relations or operations in which a mathematician is likely to actually be interested. We begin with the formal definition of persistence of computability over presentations and a counterexample to our conjecture. We then show in what sense it is essentially correct.

**Definition 2.1** (Ash and Nerode [1981]) *If $R \subseteq A^n$, then $R$ is* intrinsically computable (c.e.) *if $f[R]$ is computable (c.e.) for every isomorphism $f : \mathcal{A} \to \mathcal{B}$.*

**Example 2.2** $\langle N, \leq \rangle$: The successor function is not intrinsically computable by Proposition 1.6.

**Example 2.3** $\langle N, s \rangle$: Every computable relation is intrinsically computable. Suppose $R \subseteq N$ is computable and $f : N \to B$ is another computable presentation of $N$. By the argument of Example 1.5, $f$ is computable. Of course, $\bar{R}$, the complement of $R$ in $A$, is also computable. As $f$ is a bijection $f[R]$ and $f[\bar{R}]$ are complementary in $\mathcal{B}$ . As they are both the images of computable sets under a computable map they are computably enumerable and so actually computable. (The argument for $n$-ary relations $R$ is essentially the same and we will frequently consider only unary relations when there is no real difference.)

On the other hand, computable categoricity does not guarantee that all computable relations are intrinsically computable.

**Example 2.4** Consider the structure $\mathbb{Q}$ with the usual ordering and then additional dense subsets $R$ and $R'$ with $R$ computable and $R'$ not. It is easy to construct an isomorphism $f : \langle \mathbb{Q}, \leq, R \rangle \to \langle \mathbb{Q}, \leq, R' \rangle$ and so $R$ is not intrinsically computable even though $\mathbb{Q}$ is computably categorical.

There are two views of what has gone wrong in this counterexample. One is that, although there is some computable map between the two presentations of $\mathbb{Q}$ given in this example (indeed the identity map is such an isomorphism as $R$ is not in the language), the isomorphism $f$ is not computable. The way to solve this problem is to strengthen the notion of computable categoricity.

**Definition 2.5** *$\mathcal{A}$ is* computably stable *if every isomorphism $f : \mathcal{A} \to \mathcal{B}$ is computable.*

Thus the Examples above show that $\langle N, s \rangle$ is computably stable but $\langle \mathbb{Q}, \leq \rangle$, while computably categorical, is not computably stable. The stronger notion, however, clearly suffices to guarantee that the computability of any relation cannot change from one presentation to another. Perhaps surprisingly, it is easy to see that it is also necessary.

**Proposition 2.6** (Ash and Nerode [1981]) *$\mathcal{A}$ is computably stable if and only if every computable relation on $\mathcal{A}$ is intrinsically computable.*

**Proof.** The argument given in Example 2.3 proves sufficiency. For necessity, suppose every computable relation on $\mathcal{A}$ is intrinsically computable and consider the successor relation $R$ on $A$ as a computable set of natural numbers with the usual ordering.

Suppose $\mathcal{B}$ is any other presentation of $\mathcal{A}$ given by $f : A \to B$. As the image of $R$ is computable we can use it to computably calculate $f$ as in Example 1.5. $\Box$

The other view of what is wrong with Example 2.4 is that the relation $R$ considered is not invariant under automorphisms and so not mathematically relevant to the structure. This suggest the following proposition saying that computable categoricity is enough to guarantee intrinsic computability for all the relations of interest.

**Proposition 2.7** *If $\mathcal{A}$ is computably categorical then every invariant (under automorphisms) computable relation on $\mathcal{A}$ is intrinsically computable.*

**Proof.** Suppose $\mathcal{A}$ is computably categorical, $R \subseteq A$ is computable, and $g$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. We wish to show that $g[R]$ is computable. As $\mathcal{A}$ is computably categorical, there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$. As in Example 2.3, $f[R]$ is computable. As $R$ is invariant under automorphisms, in particular under $g^{-1}f$, $f[R] = g[R]$ and so $g[R]$ is also computable. $\Box$

Before considering specific conditions on structures that imply computable categoricity or stability and so guarantee that computational properties do not vary among presentations, we mention a number of general results on when various specified types of mathematical structures are computably categorical.

**Theorem 2.8** (Goncharov [1973]; LaRoche [1977]; Remmel [1981]; Goncharov and Dzgoev [1980]) *A Boolean algebra is computably categorical if it has finitely many atoms. If not, it has dimension $\omega$.*

**Theorem 2.9** (Remmel [1981a]; Goncharov and Dzgoev [1908]) *A linear order is computably categorical if it has only finitely many adjacent pairs of elements. If not, it has dimension $\omega$.*

**Theorem 2.10** (Nurtazin [1974]; Metakides and Nerode [1979]) *A real or algebraically closed field of finite transcendence degree over $\mathbb{Q}$ is computably categorical. Ones of infinite degree have dimension $\omega$.*

**Theorem 2.11** (Goncharov [1980]) *Every Abelian group has dimension $1$ or $\omega$.*

**Theorem 2.12** (Goncharov, Lempp and Solomon [2000]) *Every Archimedian ordered group has dimension $1$ or $\omega$.*

These results provide a type of dichotomy theorem for each of the classes of structures described. Every one is either computably categorical and so very well behaved in terms of computability properties or has the maximum possible number of computable isomorphism types. Moreover, in each case, there is a structural analysis that tell us into which category each model falls. As is indicated by these examples, when there is such a dichotomy the dividing line seems to be that some sort of finiteness property characterizes the computably categorical structures. We saw this explicitly in the analysis of $\langle N, s\rangle$ in Example 1.5 where the structure's being finitely generated in the usual sense was the key to computable stability. Of course, any finitely generated computable structure is computably stable by the same argument. As it turns out, with the proper version of what it means to be finitely generated and an additional assumption on the decidability of what for algebras amounts to the existence of solutions of systems of equations and inequations, this condition is also necessary.

**Definition 2.13** *A structure $\mathcal{A}$ is* $n$-decidable *(for $n \in \mathbb{N}$) if the set of prenex sentences of $Th(\mathcal{A}, a)_{a\in A}$ with $n-1$ alternations of quantifiers is computable. So, for example, $\mathcal{A}$ is* 1-decidable *if the set of prenex sentences of $Th(\mathcal{A}, a)_{a\in A}$ with either only existential or only universal quantifiers is decidable.*

**Theorem 2.14** (Ash and Nerode [1981]; Goncharov [1975]) *If $\mathcal{A}$ is 1-decidable, then $\mathcal{A}$ is computably stable if and only if there are constants $\overline{c} \in A$ and a computable sequence $\phi_i(\overline{c}, x)$ of existential formulas such that for each $i$ there is a unique $a \in A$ satisfying $\phi_i$ and each $a \in A$ satisfies some $\phi_i$.*

Here the sequence of formulas $\phi_i$ provide the generating process from the finitely many parameters $\bar{c}$ as we can search for witnesses to the existential formulas to generate the unique element $a$ satisfying each $\phi_i$. This is situation for $\mathbb{Q}$ in Example 1.7 where the formulas express the facts that each element is either a sum of 1's or the solution $r$ of an equation of the form $p + r = 0$ or $p = r \times q$ where $p$ is a sum of 1's and $q$ is such a sum or a solution to an equation of the first type. On the other hand, $\bar{\mathbb{Q}}$ is not finitely generated in the usual sense and the argument that it is computably categorical relies on specifying elements as roots of polynomials over $\mathbb{Q}$ and so determining them only up to isomorphism. Given the previously mentioned relationship between computable categoricity and the intrinsic computability of invariant relations, it is not unreasonable that we have a result for computable categoricity that is analogous to Theorem 2.14.

**Theorem 2.15** (Goncharov [1975]) *If $\mathcal{A}$ is 2-decidable then it is computably categorical if and only if there are $\overline{c} \in A$ and $\phi_i(\overline{c}, x)$ as in Theorem 2.14 except that the $\phi_i$ determine elements not uniquely but only up to automorphism, i.e. if $\phi_i(\overline{c}, a)$ and $\phi_i(\overline{c}, b)$ then there is an automorphism of $\mathcal{A}$ taking $a$ to $b$.*

The decidability assumptions in Theorem 2.14 and 2.15 are necessary (Goncharov [1977]). Moreover, the type of dichotomy results given in Theorems 2.8-2.12 are also provided by decidability type conditions on the structures themselves.

**Theorem 2.16** (Goncharov [1982]; Nurtazin [1974], Goncharov [1977]) *Every $\Delta_2$-categorical structure (i.e. there is always a $\Delta_2$ isomorphism between any two presentations) and every $1$-decidable structure has $\dim 1$ or $\omega$.*

However, the dichotomy between computable categoricity and dimension $\omega$ does not hold for all types of structures. The first indication of such phenomena was in the following theorem:

**Theorem 2.17** (Goncharov [1980a]) *For each $n$, $1 \leq n \leq \omega$ there is a structure of dimension $n$.*

Indeed, there are several examples of common mathematical theories whose models can have all possible dimensions and, as we shall see in Theorem 4.1, exhibit the widest possible range of variation in computability properties.

In addition to trying to characterize when a structure is computably categorical or stable, it is natural to ask when individual relations are intrinsically computable. As with structures, there is a nice answer if we assume a certain amount of decidability.

**Theorem 2.18** (Ash and Nerode [1981]) *If $\langle \mathcal{A}, R \rangle$ is 1-decidable then $R$ is intrinsically c.e. if and only if, for some finite list of parameters $\vec{a}$ from $A$ and computable set of existential formulas $\phi_i$, $\mathcal{A} \models R(\vec{x}) \Leftrightarrow \bigvee \phi_i(\vec{x}, \vec{a})$. Of course, $R$ is intrinsically computable if and only if it and its complement are both intrinsically c.e.*

The possible routes to the failure of computable categoricity or stability or intrinsic computability are intimately connected with the possible complexity of a given computable relation $R$ on $\mathcal{A}$ in various presentations of $\mathcal{A}$. We explore this topic in the next section.

# 3 Degree Spectra

The question we want to address here is how complicated can $f[R]$ be for a computable (c.e.) relation $R$ on $\mathcal{A}$ if $R$ is not intrinsically computable. The idea of the range of variability is captured in the following definition from Harizanov [1987].

**Definition 3.1** *If $R$ is an $n$-ary relation on $\mathcal{A}$, the* degree spectrum of $R$, $DgSp(R)$, *is* $\{\deg_T(f[R]) \mid f : \mathcal{A} \to \mathcal{B}$ *is an isomorphism*$\}$. *($\deg_T(X)$ is the Turing degree of $X$.)*

Of course, $R$ is intrinsically computable if and only if $DgSp(R) = \{\mathbf{0}\}$. Many natural examples of relations that are not intrinsically computable have degree spectrum the set $\mathcal{C}$ of all c.e. degrees or $\mathcal{D}$, the set of all degrees. Again there are conditions that guarantee that $R$ has such a spectrum (Harizanov [1987], [1991]; Ash Cholak and Knight [1997]) but here we concentrate on the possibilities of producing more specific sets of degrees as degree spectra of computable relations. The general question is which sets of degrees can be realized as the degree spectrum of a computable relation. We mention a few examples.

**Theorem 3.2** (Harizanov [1993]) *There is an $\mathcal{A}$ and a relation $R$ on $\mathcal{A}$ such $\mathcal{A}$ has exactly two computable presentations and $DgSp(R) = \{\mathbf{0}, \mathbf{c}\}$ with $\mathbf{c}$ noncomputable and $\Delta^0_2$.*

**Theorem 3.3** (Khoussainov and Shore [1998]) *For any computable partially ordered set $\mathcal{P}$ (with least element) there exists a structure $\mathcal{A}$ of dimension the cardinality of $\mathcal{P}$ and a (computable) unary relation $U$ on $\mathcal{A}$ such that $DgSp(U) \cong \mathcal{P}$ and every element of $DgSp(U)$ is c.e.*

**Theorem 3.4** (Hirschfeldt [1999]) *For every uniformly c.e. set $\mathcal{W}$ of c.e. degrees. there is an $\langle \mathcal{A}, R \rangle$ with $DgSp(R) = \mathcal{W}$.*

A more difficult problem (in the finite dimensional case) is to control both the degree spectrum of $R$ and the computable dimension of $\mathcal{A}$ simultaneously. The following theorem subsumes all of the above results on finite degree spectra as well as Theorem 2.17 on computable dimension.

**Theorem 3.5** (Hirschfeldt [1999]; Khoussainov and Shore [2000]) *For every finite set $\mathcal{W}$ of c.e. degrees, there is an $\langle \mathcal{A}, R \rangle$ such that $DgSp(R) = \mathcal{W}$ and $\dim(\mathcal{A}) = |\mathcal{W}|$.*

Many natural infinite classes of degrees other than $\mathcal{D}$ and $\mathcal{C}$ can also be realized as the degree spectra of computable relations. Here are some examples.

**Theorem 3.6** (Hirschfeldt [1999]) *Each of the following classes can be realized as the degree spectrum of a computable relation: the $\Sigma^0_n$ degrees, the $\Delta^0_n$ degrees, $\mathcal{D}(\leq \mathbf{a})$ and $\mathcal{C}(\leq \mathbf{a})$ for any c.e. $\mathbf{a}$.*

The only known limitations on the degree spectrum of a computable relation $R$ are imposed by the fact that for $R \subseteq A^n$, $\{f[R]|$ there is a computable $\mathcal{B}$ and an isomorphism $f : \mathcal{A} \to \mathcal{B}\}$ is $\Sigma^1_1$ in $R$. Thus, there are countable partial orders that cannot be realized in the c.e. degrees as the degree spectrum of any relation $R$ on any structure $\mathcal{A}$. Similarly, such a partial order with least element cannot be realized anywhere in the Turing degrees as the degree spectrum of a computable relation. Indeed, not every finite set of degrees can be realized as a degree spectrum: Any degree spectrum containing a hyperarithmetic degree and a nonhyperarithmetic degree is uncountable and any unbounded spectrum contains a cone of degrees.

It turns out that controlling the degree spectrum is also connected to another long standing problem involving computable categoricity. In classical model theory, a countably categorical structure remains countably categorical when expanded by finitely many constants. Results of Goncharov [1975] and Millar [1986] showed that computably categorical structures remain computably categorical when expanded by a constant if they are sufficiently decidable. (Goncharov assumed 2-decidability and Millar reduced this to 1-decidability.) The question of whether the decidability assumption is necessary was finally answered in Cholak, Goncharov, Khoussainov and Shore [1999].

**Theorem 3.7** (Cholak, Goncharov, Khoussainov and Shore [1999]) *For each $k \in \omega$, $k \geq 2$, there is a computably categorical $\mathcal{B}$ whose expansion by any constant has computable dimension $k$.*

This result was then later derived as an easy corollary of the work in Khoussainov and Shore [1998] on degree spectra. Both analyses left open the question of whether expansion by a constant could change a computably categorical structure to one with infinite dimension. Hirschfeldt's [1999] proof of Theorem 3.5 supplied the techniques to answer this question as well.

**Theorem 3.8** (Hirschfeldt, Khoussainov and Shore [2000]) *There exists a computably categorical structure $\mathcal{A}$ whose expansion by any constant has dimension $\omega$.*

# 4 Interpretations and Algebraic Examples

The original examples of structures with finite dimension and relations with finite degree spectra were always ad hoc constructions of families of computably enumerable sets or graphs. In the case of finite dimensionality, there was a sequence of results showing that this phenomena occurred in various classes of mathematical structures: partial orders and (implicitly) graphs in Goncharov [1980a] from which the case for lattices follows easily; groups and then two step nilpotent groups in Goncharov [1981] and Goncharov, Molokov and Romanovskii [1989]; and integral domains in Kudinov [1997]. Typically, these proofs proceeded by coding families of c.e. sets into structures in the desired classes. A more general coding method has now been devised for producing such results as well as many other phenomena involving degree spectra, categoricity and other computability issues in many classes of mathematical structures.

**Theorem 4.1** (Hirschfeldt, Khoussainov, Shore and Slinko [2000]) *For each of the following theories and each set $\mathcal{W}$ of c.e. degrees of size $n$, there is a model $\mathcal{A}$ and a relation $R$ on $\mathcal{A}$ (which can be taken to be a substructure of $\mathcal{A}$) such that the dimension of $\mathcal{A}$ is $n$ and the degree spectrum of $R$ is $\mathcal{W}$. There is also a model $\mathcal{B}$ which is computably categorical but some expansion by a constant has dimension $n$ and one which has dimension $\omega$ when expanded by a constant: graphs, lattices, partial orders, commutative semigroups, rings (with zero divisors), integral domains and nilpotent groups.*

The analysis here begins with classical interpretability methods for theories as in Hodges [1993] and takes as its starting structures graphs (which are just symmetric irreflexive binary relations). Suppose we want to interpret arbitrary graphs $\mathcal{G}$ in a class $\mathfrak{A}$ of structures $\mathcal{A}$. The standard procedure calls for formulas $\phi_D(x)$ and $\phi_R(x,y)$ in the language of $\mathcal{A}$ that specify a domain $D^{\mathcal{A}}$ and an edge (SIB) relation $R^{\mathcal{A}}$ on $D^{\mathcal{A}}$ so that for each graph $\mathcal{G}$ there is an $\mathcal{A} \in \mathfrak{A}$ such that $\mathcal{G} \cong \langle D^{\mathcal{A}}, R^{\mathcal{A}} \rangle$. For our purposes we need especially effective translations. $D$ and $R$ must be intrinsically computable with a computable "natural transformation" from $D^{\mathcal{A}}$ to $\mathcal{G}$. In this way we can preserve computability properties across the interpretation. To deal with issues of isomorphisms, $D$ and $R$ must be invariant. To control dimension, we also want the interpreted structure $\mathcal{A}_{\mathcal{G}} = (D^{\mathcal{A}}, R^{\mathcal{A}})$ to determine $\mathcal{A}$ in an effective way and any isomorphism of $\mathcal{A}_{\mathcal{G}}$ to be extendible to one of $\mathcal{A}$. We guarantee this last condition by requiring that there be, for each $\mathcal{G}$, a computable family $\phi_i(\bar{a}, \bar{b}_i, x)$ of existential formulas with $\bar{a} \in A$ and $\bar{b}_i \in D^{\mathcal{A}}$ such that every $x \in \mathcal{A}_{\mathcal{G}}$ satisfies some $\phi_i(\bar{a}, \bar{b}_i, x)$ and no two distinct elements of $D^{\mathcal{A}}$ satisfy any one $\phi_i(\bar{a}, \bar{b}_i, x)$.

These conditions on interpretations suffice for the simplest of the classes mentioned in the theorem. For rings and integral domains we must generalize to interpreting equality as a new (invariant, intrinsically computable) equivalence relation $Q^{\mathcal{A}}$ and extend all the new conditions to deal with sets of representatives. The result for commutative semigroups follows from the interpretation into integral domains by considering a subset of the integral domain that forms a commutative semigroup under the multiplication operation of the ring. The interpretation into nilpotent groups is a two step one. The first uses the translation of graphs into integral domains (of finite characteristic $p > 2$) and the second uses Malcev's [1965] representation of a ring $R$ as the center of the group of matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \text{with } a, b, c \in R$$

with the ring operations definable from the matrix multiplication in an appropriately effective way.

The especially effective interpretations produced to prove Theorem 4.1 can be used to transfer other phenomena from arbitrary computable structures to ones of the types listed in the theorem. We give two such applications.

**Theorem 4.2** (Slaman [1998]; Wehner [1998])*: There is a structure $\mathcal{A}$ without a computable presentation that has presentations of every nonzero degree.*

**Corollary 4.3** *There are such lattices, partial orders, commutative semigroups, rings, nilpotent groups, integral domains.*

**Theorem 4.4** (Ash [1986])*: For each even computable ordinal $\alpha$ there is a well-ordering $\mathcal{W}$ which is $\Delta_{\alpha+1}$-categorical but not $\Delta_\alpha$-categorical, i.e. if $\mathcal{V}$ is a computable presentation of $\mathcal{W}$ then there is an isomorphism between $\mathcal{W}$ and $\mathcal{V}$ which is $\Delta_{\alpha+1}$ but there is some presentation $\mathcal{V}$ for which there is no such isomorphism which is $\Delta_\alpha$.*

**Corollary 4.5** *There are such lattices, partial orders, commutative semigroups, rings, nilpotent groups, integral domains.*

We close this paper by mentioning some diverse topics within mathematical logic that are related to the methods and goals of the analysis of computability properties of relations on computable structures discussed here. First, there are clear analogies between the methods and results on intrinsic computability and degree spectra and those of reverse mathematics. This subject systematically analyzes which existence assumption axioms are needed to prove standard theorems of classical mathematics. (The best source for Reverse Mathematics is Simpson [1999] although the connections with computable model theory and algebra are not always made explicit.) The fact that some relation on a structure such as "commuting with every element of a group" is not intrinsically computable typically means that the existence of the set of such elements, the center of the group, is not provable in $\mathrm{RCA}_0$ the standard base theory of reverse mathematics. The fact that the spectrum of the relation includes $\mathbf{0}'$, typically means that the proof of existence requires $\mathrm{ACA}_0$ (arithmetic comprehension). The other standard systems of reverse mathematics have similar computability theoretic analogs and usually proofs from one area carry over to the analogous ones in the other.

Second, the phenomena of dichotomies between structure theory and nonstructure theorems has become commonplace in both model theory and descriptive set theory. Of course, each area has it's own version of what one requires to have a structure theorem. Usually, the requirements from the computability standpoint are the most stringent. In model theory, the relevant issues include ones about transferring model-theoretic phenomena from structures of one class such as graphs where certain properties are easy to arrange to others such as groups where they are less obvious (as in Mekler [1981]). In descriptive set theory, the analysis frequently centers around the issue of completeness of various properties at different levels of a hierarchy with respect to Borel reducibilities (as, for example, in Friedman and Stanley [1989]; Camerlo and Gao [2000]; Hjorth and Kechris [1996]). Thus the types of interpretations described in this section for translating results on degree spectra will supply ones that can be used in both model theory and descriptive set theory. On the other hand, even when not entirely effective the work in these other areas can at times be used in our setting. One example is the interpretations of graphs in fields in Friedman and Stanley [1989]. Although not quite satisfying the conditions we need in Theorem 4.1, their interpretation can be used, for example, to derive the analog of Theorem 4.4 for fields as long as $\alpha$ is sufficiently large. We look forward to the development of further connections among all these topics.

## 5 Bibliography


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