# Solutions of the Goncharov-Millar and Degree Spectra Problems in The Theory of Computable Models


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The theory of computable models is an intensively developing area of mathematics that studies the interactions between the theory of models and computability theory. Analyzing the relationships between computable presentations of models, model-theoretic definability and the computable complexity of relations is one of the central problems in this area. A fundamental notion in the study of these problems is that of being computable isomorphic or autoequivalent as first introduced by A.I. Malcev [5]. We call a model $\mathcal{B}$ *computable* if its domain, basic predicates and operations are uniformly computable. If a model $\mathcal{B}$ is computable and is isomorphic to a model $\mathcal{A}$, then $\mathcal{B}$ is called a *computable presentation* of $\mathcal{A}$. Computable presentations $\mathcal{B}_1$ and $\mathcal{B}_2$ are *computably isomorphic* (*autoequivalent)* if there exists a computable isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. The maximal number of computable but not computably isomorphic presentations of a model $\mathcal{B}$ is called the *computable* (*algorithmic*) *dimension* of $\mathcal{B}$ and is denoted by $dim(\mathcal{B})$. A model $\mathcal{B}$ is *computably categorical* (*autostable*) if $dim(\mathcal{B}) = 1$. Atomless Boolean algebras and dense linearly ordered sets are typical examples of computably categorical models. The notion of computable dimension was introduced by Goncharov. He proved that for any natural number $n \geq 1$ there exists a model whose computable dimension is $n$ [2]. By an appropriate coding of these models of Goncharov, examples of groups, partially ordered sets, unary and other algebras of computable dimension $n$ have been constructed in [2] [3] [6].

One of the important problems in this area is that of Goncharov-Millar about the relationship between the computable dimension of any given model $\mathcal{B}$ that of its expansion $(\mathcal{B}, c_1, \ldots, c_m)$ by finitely many constants. We note that the following inequality always holds: $dim(\mathrm{B}) \leq dim(\mathrm{B}, c_1, \ldots, c_m)$. The following theorem gives a full solution to the Goncharov-Millar problem.

**Theorem 1** *For any nonzero cardinal $n \leq \omega$ there exists a computably categorical model $\mathcal{B}$ such that the computable dimension of the model $(\mathcal{B}, c)$ is $n$ for any given constant $c \in B$.*


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For finite $n$, this theorem is proved in [7]. The case $n = \omega$ requires some new ideas and a new mechanism for constructing the desired model. It has been proved by the authors in collaboration with D. Hirschfeldt. The following is a corollary to the proof of the theorem.

**Corollary 2** *1. For all $n, m \in \omega$ there exists a model $\mathcal{B}$ such that $dim(B) = n + 1$ and $dim(B, c) = n + m + 1$ for any constant $c \in \mathcal{B}$.*

*2. For all $n \in \omega$ there exists a model $\mathcal{B}$ such that $dim(B) = n + 1$ and $dim(B, c) = \omega$ for any constant $c \in \mathcal{B}$.*

One of the central problems directly related to the study of computable dimension is the problem of characterizing the computable complexity of a relation (which is not included in the language) in computable presentations of a model. This problem was informally stated by Nerode at the beginning of the 70s. In order to study this problem, Harizanov and Millar [9] introduced the notion of the degree spectrum of a given relation. The *spectrum* of a relation $R$ on a model $\mathcal{B}$, denoted by $S(R)$, is the set of all Turing degrees of images of $R$ in computable presentations of the model $\mathcal{B}$. The study of this set is closely related to questions about the definability of $R$ in the language $L_{\omega_1,\omega}$. It has been extensively studied by Ash, Knight, Nerode and others. It is obvious that if $dim(\mathcal{B}) < \omega$ and $R$ is fixed by each automorphism of the model, then the set $S(R)$ is finite. Harizanov noted in [9] that Goncharov's model of computable dimension 2 provides an example of a relation $R$ such that $S(R) = \{0, \mathbf{a}\}$, where 0 is the Turing degree of the recursive sets and $\mathbf{a}$ is the Turing degree of a set which is $\Delta^0_3$ in the Kleene-Mostowski hierarchy. Using the construction from [1], she constructed in [9] a model $\mathcal{B}$ with a relation $R$ such that $dim(\mathrm{B}) = 2$ and $S(R) = \{0, \mathbf{a}\}$, where $\mathbf{a}$ is the degree of a $\Delta^0_2$-set. Goncharov and Khoussainov [4] improved this result by constructing a model $\mathcal{B}$ with a relation $R$ such that $dim(\mathrm{B}) = 2$ and $S(R) = \{0, \mathbf{a}\}$, where $\mathbf{a}$ is the degree of a recursively enumerable set. However, the following two questions, formulated in [4], that constitute the *degree spectra problem* had remained opened.

**Question 1** *Which finite partially ordered sets are isomorphic to partially ordered sets of the type $(S(R), \leq)$, where $\leq$ is Turing reducibility?*

**Question 2** *Which finite sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of recursively enumerable degrees coincide with $S(R)$, where $R$ is a relation on a model whose computable dimension is $n$?*

In [8] the authors give a full solution to Question 1. Very recently the authors have also been able to answer the second question.

**Theorem 3** *For any finite set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of recursively enumerable degrees there exists a model $\mathcal{B}$ and a relation $R$ on it such that $dim(B) = n$ and $S(R) = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.*

This theorem also answers the first question because any countable partially ordered set can be embedded into the set of recursively enumerable Turing degrees. We note that D. Hirschfeldt has also recently proved this theorem independently using somewhat different ideas [10].

The main step in our proof of Theorem 2 consists of a construction that builds a model $\mathcal{B}$ and a relation $R$ on it with the following properties:

1. $dim(\mathcal{B}) = 2$,
2. $S(R) = \{0, \mathbf{a}\}$,

where **a** is the Turing degree of a given recursively enumerable set. In order to satisfy these properties the construction must build two computable presentations $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$, isomorphic relations $R_1$ and $R_2$ on $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, satisfying the following requirements:

$D_e$) $\phi_e$ is not an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$,

$R_j$) If $\mathcal{C}_j$ is isomorphic to $\mathcal{B}$, then $C_j$ is computably isomorphic to either $\mathcal{B}_1$ or $\mathcal{B}_2$,

$P_1$) The set $R_1$ is recursive,

$P_2$) The set $R_2$ is recursively enumerable and of degree **a**,

where **a** is a given recursively enumerable degree; $\phi_e$, $e \in \omega$, is an enumeration of all partial recursive functions; and $\mathcal{C}_j$, $j \in \omega$, is an enumeration of all recursively enumerable models. In order to satisfy all these requirements, two problems must be solved. The first problem arises from the conflicts between the requirements $D_e$ and $R_j$, $e, j \in \omega$. The resolution of these conflicts allows us to control the computable dimension of $\mathcal{B}$. The ideas for controlling the computable dimension of $\mathcal{B}$ come from [8] and [1]. The second problem arises in resolving the conflict between controlling the computable dimension of $\mathcal{B}$ and coding the degree **a** into $R_2$. Two ideas are used in the solution of this problem. First, the model $\mathcal{B}$ is taken from a special class of graphs so that algebraic properties, in particular, various connectedness properties of $\mathcal{B}$ influence the construction's outcomes at certain stages. Second, the requirements $D_e$, $e \in \omega$, together with the coding requirement $P_2$, are satisfied by using a new procedure for modifying the graph being constructed as numbers are enumerated into a fixed set A of degree **a** so as to code A into the graph.

Now, all known computable models $\mathcal{B}$ of finite computable dimensions, if not computably categorical, are at least $\Delta^0_3$–categorical, i.e. if $\mathcal{A}$ is computable and isomorphic to $\mathcal{B}$ then there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ which is $\Delta^0_3$. We conclude this paper with the following question.

**Question 3** *Is there, for each $n \geq 3$, a computable model of computable dimension* 2 *which is $\Delta^0_{n+1}$-categorical but not $\Delta^0_n$-categorical?*

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