Mass problems and density*

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Abstract

Recall that $\mathcal{E}_{\mathrm{w}}$ is the lattice of Muchnik degrees of nonempty effectively compact sets in Euclidean space. We solve a long-standing open problem by proving that $\mathcal{E}_{\mathrm{w}}$ *is dense*, i.e., satisfies $\forall x\,\forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$. Our proof combines an oracle construction with hyperarithmetical theory.

1 Introduction

The study of degrees of unsolvability began in the mid-20th century. Given a set A of natural numbers, there is an associated *decision problem*, the problem of "deciding" for each natural number n whether $n \in A$ or not. As a method of measuring the algorithmic unsolvability of decision problems, Kleene and Post [10] introduced the semilattice $\mathcal{D}_{\mathrm{T}}$ consisting of the *Turing degrees* of arbitrary sets A, while Post [15] emphasized the countable subsemilattice $\mathcal{E}_{\mathrm{T}}$ consisting of the *recursively enumerable Turing degrees*, i.e., Turing degrees of sets A which are recursively enumerable. The semilattices $\mathcal{D}_{\mathrm{T}}$ and $\mathcal{E}_{\mathrm{T}}$ are defined in terms of *Turing oracles*, a notion which had been introduced earlier by Turing [30, §4]. Two important landmarks in the study of $\mathcal{E}_{\mathrm{T}}$ were Sacks's discovery of the *splitting theorem* [17]:

> Every nonzero recursively enumerable Turing degree is the supremum of two smaller recursively enumerable Turing degrees.

and the *density theorem* [18]:

> For any two comparable recursively enumerable Turing degrees, there is another recursively enumerable Turing degree between them.

Inspired by these results of Sacks, the study of structural properties of $\mathcal{E}_{\mathrm{T}}$ has played a leading role in recursion theory from the 1960s to the present. See for instance the recent paper [1] and the survey papers [12, 22].

Also in the mid-20th century but largely ignored in the West, Medvedev [13] and Muchnik [14] introduced more general notions of degrees of unsolvability. Given a set P in Euclidean space or some similar space such as

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the Cantor space $\{0,1\}^{\mathbb{N}}$ or the Baire space $\mathbb{N}^{\mathbb{N}}$, there is an associated *mass problem*[1], the problem of "finding" or "computing" some element of P. Thus P plays the role of the "solution set" of the mass problem. As a method of measuring the algorithmic unsolvability of mass problems, Medvedev [13] and Muchnik [14] introduced the lattices $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ of *strong degrees* and *weak degrees*, also known as *Medvedev degrees* and *Muchnik degrees* respectively. The idea here is that a mass problem P is reducible to a mass problem Q if, given any solution of Q, we can use it as a Turing oracle to compute some solution of P. By requiring the computation to be uniform, we get strong reducibility, a.k.a., Medvedev reducibility. By allowing the computation to be nonuniform, we get weak reducibility, a.k.a., Muchnik reducibility. See also the formal definition of $P \leq_{\mathrm{w}} Q$ in §2 below.

At the end of the 20th century, and in ignorance of $\mathcal{D}_{\mathrm{w}}$ but motivated by a desire to highlight certain aspects of $\mathcal{E}_{\mathrm{T}}$, Simpson [23, 24] introduced the countable sublattice $\mathcal{E}_{\mathrm{w}}$ of $\mathcal{D}_{\mathrm{w}}$ consisting of the Muchnik degrees of nonempty sets which are *effectively compact*, i.e., Π^0_1 and recursively bounded. The study of $\mathcal{E}_{\mathrm{w}}$ continued in a number of 21st century publications including the survey papers [27, 28].

As noted in [23, 24] and further emphasized in [25, 26, 28], there is a compelling analogy between $\mathcal{E}_{\mathrm{T}}$ and $\mathcal{E}_{\mathrm{w}}$:

1. With respect to the *arithmetical hierarchy* [16, Chapters 14 and 15], $\mathcal{E}_{\mathrm{w}}$ is the smallest nontrivial sublattice of $\mathcal{D}_{\mathrm{w}}$, just as $\mathcal{E}_{\mathrm{T}}$ is the smallest nontrivial subsemilattice of $\mathcal{D}_{\mathrm{T}}$.

2. By [26] we have a natural embedding of $\mathcal{E}_{\mathrm{T}}$ into $\mathcal{E}_{\mathrm{w}}$, given by $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1})$ where $\mathbf{1}$ is the top degree in $\mathcal{E}_{\mathrm{w}}$. This embedding is one-to-one and preserves the algebraic structure of $\mathcal{E}_{\mathrm{T}}$ including the top and bottom degrees, the reducibility ordering, and the semilattice operation.

3. The splitting and density theorems hold for $\mathcal{E}_{\mathrm{w}}$, just as they do for $\mathcal{E}_{\mathrm{T}}$. The splitting theorem for $\mathcal{E}_{\mathrm{w}}$ is due to Binns [4]. The density theorem for $\mathcal{E}_{\mathrm{w}}$ is the main theorem of this paper.

On the other hand, $\mathcal{E}_{\mathrm{w}}$ seems to have a significant advantage over $\mathcal{E}_{\mathrm{T}}$:

1.

[1] The term "mass problem" is a literal translation of the Russian term in [13] and was used by Rogers in his seminal textbook [16].

4. A great many specific natural degrees have been discovered in $\mathcal{E}_{\mathrm{w}}$. See for instance [28, Figure 1]. By contrast, no specific natural degrees in $\mathcal{E}_{\mathrm{T}}$ are known except the top degree $\mathbf{0}'$ and the bottom degree $\mathbf{0}$. The problem of finding other specific natural degrees in $\mathcal{E}_{\mathrm{T}}$ remains open, despite more than 50 years of intensive research on structural aspects of $\mathcal{E}_{\mathrm{T}}$.

We feel that these considerations help to motivate and justify the study of $\mathcal{E}_{\mathrm{w}}$.

As already mentioned, the main result of this paper is the density theorem for $\mathcal{E}_{\mathrm{w}}$. This answers a question which was implicit in [3, 4, 5] and explicit in [21, last paragraph] and [26, Remark 2.11] and [27, Remark 3.1.3].

The paper is organized as follows. In §2 we set up some notation. In §3 we present our density proof. We end with some open questions.

2 Essential notation and definitions

In this section we develop notation and definitions which are needed for precise understanding of the statements of our results in §3.

The set of nonnegative integers is denoted $\mathbb{N}$. Variables such as i, j, m, n, s, t range over $\mathbb{N}$. The set of all functions $X : \mathbb{N} \to \mathbb{N}$ is denoted $\mathbb{N}^{\mathbb{N}}$. Variables such as $X, Y, Z, \ldots$ range over $\mathbb{N}^{\mathbb{N}}$. We write $\{n\}^Y(i) = j$ to mean that the Turing machine with Gödel number n using Y as a Turing oracle started with input i eventually halts with output j. We write $\{n\}^Y(i)\downarrow$ if $\exists j\,(\{n\}^Y(i) = j)$, otherwise $\{n\}^Y(i)\uparrow$. We say that X is *Turing reducible* to Y, abbreviated $X \leq_{\mathrm{T}} Y$, if $\exists n\,\forall i\,(X(i) = \{n\}^Y(i))$. It is known that $\leq_{\mathrm{T}}$ is reflexive and transitive. We say that X is *Turing equivalent* to Y, abbreviated $X \equiv_{\mathrm{T}} Y$, if $X \leq_{\mathrm{T}} Y$ and $Y \leq_{\mathrm{T}} X$. The *Turing jump* of $X \in \mathbb{N}^{\mathbb{N}}$ is $X' \in \mathbb{N}^{\mathbb{N}}$ defined by letting $X'(n) = 1$ if $\{n\}^X(n)\downarrow$, otherwise $X'(n) = 0$.

For $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$ we say that P is *Muchnik reducible* to Q, abbreviated $P \leq_{\mathrm{w}} Q$, if $\forall Y\,(Y \in Q \Rightarrow \exists X\,(X \in P \text{ and } X \leq_{\mathrm{T}} Y))$. Clearly $\leq_{\mathrm{w}}$ is reflexive and transitive. We say that P is *Muchnik equivalent* to Q, abbreviated $P \equiv_{\mathrm{w}} Q$, if $P \leq_{\mathrm{w}} Q$ and $Q \leq_{\mathrm{w}} P$. The *Muchnik degree* of P, denoted $\deg_{\mathrm{w}}(P)$, is the equivalence class of P under $\equiv_{\mathrm{w}}$. We define $\deg_{\mathrm{w}}(P) \leq \deg_{\mathrm{w}}(Q)$ to mean that $P \leq_{\mathrm{w}} Q$. Let $\mathcal{D}_{\mathrm{w}}$ be the set of all Muchnik degrees, partially ordered by $\leq$. It is known that $\mathcal{D}_{\mathrm{w}}$ is a complete and completely distributive lattice. A set $Q \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be Π^0_1 or *effectively closed* if

$Q = \{Y \mid \{n\}^Y(n)\uparrow\}$ for some n. For $X, Y \in \mathbb{N}^\mathbb{N}$ we define $X \oplus Y \in \mathbb{N}^\mathbb{N}$ by the equations $(X \oplus Y)(2i) = X(i)$, $(X \oplus Y)(2i+1) = Y(i)$ for all i. A binary predicate $U \subseteq \mathbb{N} \times \mathbb{N}^\mathbb{N}$ is said to be Π^0_1 if the set $\{X \oplus Y \mid U(X, Y)\}$ is Π^0_1.

As in the survey paper [27], let $\mathcal{E}_\mathrm{w}$ (respectively $\mathcal{S}_\mathrm{w}$) be the lattice of Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^\mathbb{N}$ (respectively $\mathbb{N}^\mathbb{N}$).[2] It is easy to see that $\mathcal{E}_\mathrm{w}$ and $\mathcal{S}_\mathrm{w}$ are countable sublattices of $\mathcal{D}_\mathrm{w}$. It is also known that $\mathcal{E}_\mathrm{w}$ is an initial segment of $\mathcal{S}_\mathrm{w}$. For a proof of this important fact, see [26, Lemma 3.3] or [27, Theorem 3.3.1, Corollary 3.3.4].

3 Density proof

In this section we prove that $\mathcal{S}_\mathrm{w}$ and consequently $\mathcal{E}_\mathrm{w}$ are *dense*, i.e., they satisfy the sentence $\forall x\, \forall y\, (x < y \Rightarrow \exists z\, (x < z < y))$. Our proof combines a Turing jump oracle construction with some well known facts about relative hyperarithmeticity. Our oracle construction may be compared to the previously known constructions for [7, Lemma 5.1] and [29, Lemma 3.2].

Remark 1. The relevance of hyperarithmetical theory for the general study of $\mathcal{E}_\mathrm{w}$ was already clear in [7]. However, our work in this paper represents the first time that hyperathmetical theory has been used to prove a lattice-theoretic property of $\mathcal{E}_\mathrm{w}$. We do not know how to prove the density of $\mathcal{E}_\mathrm{w}$ without using hyperarithmetical theory.

As a warm-up for our oracle construction in Lemma 2, we first prove the following lemma, which is a generalization of the Friedberg Jump Theorem [16, §13.3, Corollary IX(a)].

Lemma 1. Let $Q \subseteq \mathbb{N}^\mathbb{N}$ be Π^0_1 such that $Q \not\leq_\mathrm{w} \{0\}$. Given $Z \in \mathbb{N}^\mathbb{N}$, we can find $\widetilde{Z} \in \mathbb{N}^\mathbb{N}$ such that $0' \oplus Z \equiv_\mathrm{T} 0' \oplus \widetilde{Z} \equiv_\mathrm{T} \widetilde{Z}'$ and $Q \not\leq_\mathrm{w} \{\widetilde{Z}\}$.

Proof. Before presenting our construction, we fix some additional notation.

[2] Let d be a positive integer, and let $\mathbb{R}^d$ denote d-dimensional Euclidean space. A set in $\mathbb{R}^d$ is said to be *effectively closed* if it is the complement of a set which is *effectively open*, i.e., of the form $\bigcup_{i=0}^\infty B(a_i, r_i)$ where $\langle a_i \rangle_{i\in\mathbb{N}}$ is a recursive sequence of d-tuples of rational numbers, $\langle r_i \rangle_{i\in\mathbb{N}}$ is a recursive sequence of rational numbers, and $B(a_i, r_i) = \{x \in \mathbb{R}^d \mid |x - a_i| < r_i\}$. A set in $\mathbb{R}^d$ is *effectively compact* if and only if is bounded and effectively closed. It is well known and easy to see that a Muchnik degree belongs to $\mathcal{E}_\mathrm{w}$ if and only if it contains a nonempty effectively compact set in $\mathbb{R}^d$.

A *string* is a finite sequence of nonnegative integers. Variables such as $\rho, \sigma, \tau, \ldots$ range over strings. We write $\sigma = \langle \sigma(0), \ldots, \sigma(|\sigma|-1) \rangle$ where $|\sigma| =$ the length of σ. We write

$$\sigma^\frown\tau = \langle \sigma(0), \ldots, \sigma(|\sigma|-1), \tau(0), \ldots, \tau(|\tau|-1) \rangle$$

for the *concatenation*, σ followed by τ. Thus $|\sigma^\frown\tau| = |\sigma| + |\tau|$. We write $\sigma \subset \tau$ or $\sigma \subset Z$ to mean that σ is a proper initial segment of τ or of Z respectively. We write $\sigma \subseteq \tau$ to mean that $\sigma \subset \tau$ or $\sigma = \tau$. We write $Z{\restriction}n =$ the unique string σ of length n such that $\sigma \subset Z$. A *tree* is a set U of strings such that $\forall\sigma\,\forall\tau\,(\sigma \subset \tau \in U \Rightarrow \sigma \in U)$. A *path* through U is any $Y \in \mathbb{N}^\mathbb{N}$ such that $\forall n\,(Y{\restriction}n \in U)$. It is well known that $Q \subseteq \mathbb{N}^\mathbb{N}$ is Π^0_1 if and only if $Q = \{Y \mid Y \text{ is a path through } U\}$ for some recursive tree U.

We write $\{n\}^Z_s(i) = j$ to mean that the Turing machine with Gödel number n using Z as a Turing oracle started with input i halts after $\le s$ steps with output j. We write $\{n\}^\sigma_s(i) = j$ to mean that $\{n\}^Z_s(i) = j$ using only oracle information from $\sigma \subset Z$. We write $\{n\}^\sigma_s(i)\downarrow$ if $\exists j\,(\{n\}^\sigma_s(i) = j)$, otherwise $\{n\}^\sigma_s(i)\uparrow$. We write $\Phi_n(\sigma) = \tau$ to mean that $\forall i\,(i < |\tau| \Rightarrow \{n\}^\sigma_{|\sigma|}(i) = \tau(i))$ and $\{n\}^\sigma_{|\sigma|}(|\tau|)\uparrow$. Note that the predicates $\{n\}^\sigma_s(i) = j$ and $\{n\}^\sigma_s(i)\downarrow$ and $\Phi_n(\sigma) = \tau$ are recursive.

We now present our construction. As above, let U be a recursive tree such that $\forall Y\,(Y \in Q \Leftrightarrow Y \text{ is a path through } U)$. We follow the standard proof of the Friedberg Jump Theorem, taking additional steps to avoid computing a path through U. Given $Z \in \mathbb{N}^\mathbb{N}$ we define a sequence of strings $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_s \subseteq \tau_{s+1} \subseteq \cdots$ as follows.

Stage 0. Let $\tau_0 = \langle\rangle$ and $n_0 = 0$ and $i_0 = 1$.

Stage $s+1$. Let $n = n_s$ and proceed as follows depending on whether i_s is equal to 1, 2, or 3.

Case 1: $i_s = 1$. Let $\tau_{s+1} = \tau_s{}^\frown\langle Z(n)\rangle$ and let $n_{s+1} = n_s$ and $i_{s+1} = 2$.

Case 2: $i_s = 2$. If there exists $\tau \supseteq \tau_s$ such that $\{n\}^\tau_{|\tau|}(n)\downarrow$, let $\tau_{s+1} =$ the least such τ, otherwise $\tau_{s+1} = \tau_s$. Either way, let $n_{s+1} = n_s$ and $i_{s+1} = 3$.

Case 3: $i_s = 3$. If there exists $\tau \supset \tau_s$ such that $\Phi_n(\tau_s) \subset \Phi_n(\tau) \in U$, let $\tau_{s+1} =$ the least such τ, and let $n_{s+1} = n_s$ and $i_{s+1} = 3$. Otherwise, let $\tau_{s+1} = \tau_s$ and let $n_{s+1} = n_s + 1$ and $i_{s+1} = 1$.

This completes the construction. Clearly we have $\forall s\,(\tau_s \subset \tau_{s+3})$, so letting $\widetilde{Z} = \bigcup_s \tau_s$ we have $\widetilde{Z} \in \mathbb{N}^\mathbb{N}$. Moreover, the entire construction is uniformly $\le_{\mathrm{T}} 0' \oplus Z$ and $\le_{\mathrm{T}} 0' \oplus \widetilde{Z}$, so in particular we have $\widetilde{Z} \le_{\mathrm{T}} 0' \oplus Z$.

We claim that there are infinitely many stages s such that $i_s = 1$. Otherwise, there would be a stage s such that $n_t = n_s$ and $i_t = 3$ for all $t \geq s$. And then, letting $n = n_s$, the construction for Case 3 would produce a recursive path $\bigcup_t \Phi_n(\tau_t)$ through U, contradicting our assumption that $Q \not\leq_{\mathrm{w}} \{0\}$.

From our claim it follows that for each n there is exactly one stage s such that $n_s = n$ and $i_s = 1$. For this s we have $Z(n) = \tau_{s+1}(|\tau_s|) = \widetilde{Z}(|\tau_s|)$ by Case 1, and this shows that $Z \leq_{\mathrm{T}} 0' \oplus \widetilde{Z}$. Moreover, for this same s we have $n_{s+1} = n$ and $i_{s+1} = 2$, so by Case 2 we have $\widetilde{Z}'(n) = 1$ if and only if there exists $\tau \supseteq \tau_{s+2}$ such that $\{n\}^{\tau}_{|\tau|}(n) \downarrow$, and this shows that $\widetilde{Z}' \leq_{\mathrm{T}} 0' \oplus Z$. Furthermore, for this same s we have $n_{s+2} = n$ and $i_{s+2} = 3$, so by our claim there is exactly one stage $t \geq s+2$ such that $n_t = n$ and $i_t = 3$ and $i_{t+1} = 1$. But then by Case 3 there is no $\tau \supset \tau_t$ such that $\Phi_n(\tau_t) \subset \Phi_n(\tau) \in U$, and this shows that $Q \not\leq_{\mathrm{w}} \{\widetilde{Z}\}$. The proof is now complete. $\square$

Remark 2. In the proof of Lemma 1, the $0'$-recursive functional $Z \mapsto \widetilde{Z}$ maps any Π^0_1 set $S \subseteq \mathbb{N}^{\mathbb{N}}$ homeomorphically onto its image $\widetilde{S} = \{\widetilde{Z} \mid Z \in S\}$. This is clear from the construction, because for all stages s such that $i_s = 1$, τ_s depends only on $Z{\upharpoonright}n_s$ and conversely $Z{\upharpoonright}n_s$ can be recovered from τ_s.

Our oracle construction is embodied in the following lemma. We formulate this lemma in greater generality than is needed for the density proof.

Lemma 2. Given Π^0_1 predicates $U, V \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, we can find a Π^0_1 predicate $\widehat{V} \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that for each X with $\{Y \mid U(X,Y)\} \not\leq_{\mathrm{w}} \{X\}$ there is a homeomorphism $Z \mapsto \widehat{Z}$ of $\{Z \mid V(X,Z)\}$ onto $\{\widehat{Z} \mid \widehat{V}(X,\widehat{Z})\}$ with the properties $X' \oplus Z \equiv_{\mathrm{T}} X' \oplus \widehat{Z} \equiv_{\mathrm{T}} (X \oplus \widehat{Z})'$ and $\{Y \mid U(X,Y)\} \not\leq_{\mathrm{w}} \{X \oplus \widehat{Z}\}$.

Proof. In order to present our construction, we need some additional notation. We write $\{n\}^{X,\sigma}_s(i) = j$ to mean that $\{n\}^{X \oplus Z}_s(i) = j$ using only oracle information from X and from $\sigma \subset Z$. We write $\{n\}^{X,\sigma}_s(i) \downarrow$ if $\exists j\,(\{n\}^{X,\sigma}_s(i) = j)$, otherwise $\{n\}^{X,\sigma}_s(i) \uparrow$. We write $\Phi^X_n(\sigma) = \tau$ to mean that $\forall i\,(i < |\tau| \Rightarrow \{n\}^{X,\sigma}_{|\sigma|}(i) = \tau(i))$ and $\{n\}^{X,\sigma}_{|\sigma|}(|\tau|) \uparrow$. Note that the predicates $\{n\}^{X,\sigma}_s(i) = j$ and $\{n\}^{X,\sigma}_s(i) \downarrow$ and $\Phi^X_n(\sigma) = \tau$ are uniformly $\leq_{\mathrm{T}} X$.

We now present our construction, which is similar in many respects to the construction for Lemma 1. For each $X \in \mathbb{N}^{\mathbb{N}}$ let U^X be a uniformly X-recursive tree such that $\forall X\, \forall Y\, (U(X,Y) \Leftrightarrow Y$ is a path through $U^X)$. To each $X \in \mathbb{N}^{\mathbb{N}}$ and each string σ we associate an infinite sequence of strings $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_s \subseteq \tau_{s+1} \subseteq \cdots$ as follows.

Stage 0. Let $\tau_0 = \langle\rangle$ and $i_0 = 1$ and $n_0 = 0$.

Stage $s+1$. Let $n = n_s$. If $n \geq |\sigma|$ the construction halts and we let $\tau_t = \tau_s$ and $i_t = 0$ and $n_t = n_s$ for all $t \geq s+1$. Otherwise we proceed as follows depending on whether i_s is equal to 1, 2, or 3.

Case 1: $i_s = 1$. Let $\tau_{s+1} = \tau_s{}^\frown\langle\sigma(n)\rangle$ and let $i_{s+1} = 2$ and $n_{s+1} = n_s$.

Case 2: $i_s = 2$. If there exists $\tau \supseteq \tau_s$ such that $\{n\}^{X,\tau}_{|\tau|}(n)\downarrow$, let $\tau_{s+1} =$ the least such τ, otherwise $\tau_{s+1} = \tau_s$. Either way, let $i_{s+1} = 3$ and $n_{s+1} = n_s$.

Case 3: $i_s = 3$. If there exists $\tau \supset \tau_s$ such that $\Phi^X_n(\tau_s) \subset \Phi^X_n(\tau) \in U^X$, let $\tau_{s+1} =$ the least such τ, and let $i_{s+1} = 3$ and $n_{s+1} = n_s$. Otherwise, let $\tau_{s+1} = \tau_s$ and let $i_{s+1} = 1$ and $n_{s+1} = n_s + 1$.

This completes the construction. Note that the construction does not depend on V. Let us write $F^X_s(\sigma) = \tau_s$ and note that the function F^X_s is uniformly $\leq_{\mathrm{T}} X'$ and *monotone*, i.e., $s \leq t$ and $\rho \subseteq \sigma$ imply $F^X_s(\rho) \subseteq F^X_t(\sigma)$.

For each $X \in \mathbb{N}^\mathbb{N}$ let V^X be a uniformly X-recursive tree such that $\forall X\,\forall Z\,(V(X,Z) \Leftrightarrow Z$ is a path through $V^X)$. Consider the tree

$$\widetilde{V}^X = \{\tau \mid \exists s\,\exists\sigma\,(\sigma \in V^X \text{ and } \tau \subseteq F^X_s(\sigma))\}.$$

For each $\tau \in \widetilde{V}^X$, our construction shows that $\tau \subseteq F^X_s(\sigma)$ for some $s \leq 3|\tau|$ and some $\sigma \in V^X$ such that σ is a *substring* of τ, i.e., $\sigma = \langle\tau(j_1), \ldots, \tau(j_{|\sigma|})\rangle$ for some $j_1 < \cdots < j_{|\sigma|} < |\tau|$. Thus, in the definition of $\widetilde{V}^X$, the unbounded quantifiers $\exists s$ and $\exists\sigma$ may be replaced by bounded quantifiers. It follows that the tree $\widetilde{V}^X$ is uniformly $\leq_{\mathrm{T}} X'$. Hence, by Post's Theorem, the predicate $\widetilde{V} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ defined by

$$\widetilde{V}(X, \widetilde{Z}) \equiv \widetilde{Z} \text{ is a path through } \widetilde{V}^X$$

is Π^0_2, say $\widetilde{V}(X,\widetilde{Z}) \equiv \forall i\,\exists j\,B(X,\widetilde{Z},i,j)$ where B is a recursive predicate. We now define our Π^0_1 predicate $\widehat{V} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ by

$$\widehat{V}(X, \widetilde{Z} \oplus \widetilde{Z}^*) \equiv \forall i\,(\widetilde{Z}^*(i) = \text{the least } j \text{ such that } B(X,\widetilde{Z},i,j) \text{ holds}).$$

To prove that $\widehat{V}$ has the desired properties, let X be such that $\{Y \mid U(X,Y)\} \not\leq_{\mathrm{w}} \{X\}$, i.e., X does not compute a path through U^X. In such a situation, our construction of $\bigcup_n F^X_s(Z{\restriction}n)$ for $s = 0,1,2,\ldots$ is the straightforward relativization to X of the construction of τ_s for $s = 0,1,2,\ldots$ in the proof of Lemma 1. Thus, letting $\widetilde{Z} = F^X(Z) = \bigcup_s \bigcup_n F^X_s(Z{\restriction}n)$, we have $X' \oplus Z \equiv_{\mathrm{T}} X' \oplus \widetilde{Z} \equiv_{\mathrm{T}} (X \oplus \widetilde{Z})'$ and $\{Y \mid U(X,Y)\} \not\leq_{\mathrm{w}} \{X \oplus \widetilde{Z}\}$. Moreover, as in Remark 2, $Z \mapsto \widetilde{Z}$ is an X'-recursive homeomorphism of $\{Z \mid V(X,Z)\}$ onto $\{\widetilde{Z} \mid \widetilde{V}(X,\widetilde{Z})\}$. Now, following the definition of $\widehat{V}$, for each such Z let

$\widetilde{Z}^*(i) =$ the least j such that $B(X, \widetilde{Z}, i, j)$ holds. Then $\widetilde{Z}^* \leq_{\mathrm{T}} X \oplus \widetilde{Z}$, hence $\widehat{Z} = \widetilde{Z} \oplus \widetilde{Z}^*$ enjoys the same properties as $\widetilde{Z}$, i.e., $X' \oplus Z \equiv_{\mathrm{T}} X' \oplus \widehat{Z} \equiv_{\mathrm{T}} (X \oplus \widehat{Z})'$ and $\{Y \mid U(X,Y)\} \nleq_{\mathrm{w}} \{X \oplus \widehat{Z}\}$ and $Z \mapsto \widehat{Z}$ is an X'-recursive homeomorphism of $\{Z \mid V(X,Z)\}$ onto $\{\widehat{Z} \mid \widehat{V}(X, \widehat{Z})\}$. This completes the proof. □

Lemma 3. Suppose Kleene's O is not hyperarithmetical in X. Then, there is a nonempty Π^0_1 set $S \subseteq \mathbb{N}^{\mathbb{N}}$ such that $S \nleq_{\mathrm{w}} \{X'\}$.

Proof. The predicate $n \notin O$ is Σ^1_1, so by the Kleene Normal Form Theorem (see [16, §16.1, Corollary III] or [19, Theorem I.1.3]), let $S(n, Z)$ be a Π^0_1 predicate such that $\forall n\,(n \notin O \Leftrightarrow \exists Z\, S(n, Z))$. For some $n \notin O$ we must have $\{Z \mid S(n, Z)\} \nleq_{\mathrm{w}} \{X'\}$, because otherwise O would be arithmetical in X, hence hyperarithmetical in X. Fix such an n and let $S = \{Z \mid S(n, Z)\}$. □

Remark 3. In Lemma 3 the property $S \nleq_{\mathrm{w}} \{X'\}$ can be strengthened to say that no $Z \in S$ is hyperarithmetical in X.

The next theorem implies that $\mathcal{S}_{\mathrm{w}}$ is dense. The special case $Q = \emptyset$ says that $\mathcal{S}_{\mathrm{w}}$ has no top degree.

Theorem 1. Let P and Q be Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$ such that $P <_{\mathrm{w}} Q$. Then, we can find a Π^0_1 set $R \subseteq \mathbb{N}^{\mathbb{N}}$ such that $P <_{\mathrm{w}} R <_{\mathrm{w}} Q$.

Proof. The set $\{X \mid X \in P \text{ and } Q \nleq_{\mathrm{w}} \{X\}\}$ is arithmetical, hence Σ^1_1, so by the Gandy Basis Theorem (see [16, §16.7, Corollary XLII(a)] or [19, Theorem III.7.2]), let $X_0 \in P$ be such that $Q \nleq_{\mathrm{w}} \{X_0\}$ and Kleene's O is not hyperarithmetical in X_0. By Lemma 3 let $S \subseteq \mathbb{N}^{\mathbb{N}}$ be nonempty Π^0_1 such that $S \nleq_{\mathrm{w}} \{X_0'\}$. Apply Lemma 2 with $U(X,Y) \equiv Y \in Q$ and $V(X,Z) \equiv Z \in S$ to get a Π^0_1 predicate $\widehat{V}(X, \widehat{Z})$. Let

$$R = \{X \oplus \widehat{Z} \mid X \in P \text{ and } \widehat{V}(X, \widehat{Z})\} \cup Q.$$

Trivially R is Π^0_1 and $P \leq_{\mathrm{w}} R \leq_{\mathrm{w}} Q$. Since $X_0 \in P$ and $Q \nleq_{\mathrm{w}} \{X_0\}$, we have $Q \nleq_{\mathrm{w}} \{X_0 \oplus \widehat{Z}\}$ and $X_0 \oplus \widehat{Z} \in R$ for all $Z \in S$, so $Q \nleq_{\mathrm{w}} R$. It remains to prove that $R \nleq_{\mathrm{w}} P$. Since $X_0 \in P$, it suffices to prove that $R \nleq_{\mathrm{w}} \{X_0\}$. If $R \leq_{\mathrm{w}} \{X_0\}$, then since $Q \nleq_{\mathrm{w}} \{X_0\}$ there must exist $X \oplus \widehat{Z} \leq_{\mathrm{T}} X_0$ such that $X \in P$ and $\widehat{V}(X, \widehat{Z})$ holds. But then $Q \nleq_{\mathrm{w}} \{X\}$, hence $X' \oplus Z \equiv_{\mathrm{T}} X' \oplus \widehat{Z}$ for some $Z \in S$, hence $Z \leq_{\mathrm{T}} X_0'$, a contradiction. This completes the proof. □

The next theorem says that $\mathcal{E}_{\rm w}$ is dense.

Theorem 2. Let P and Q be nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ such that $P <_{\rm w} Q$. Then, we can find a nonempty Π^0_1 set $R \subseteq \{0,1\}^{\mathbb{N}}$ such that $P <_{\rm w} R <_{\rm w} Q$.

Proof. This follows from Theorem 1 plus the fact, noted in §2, that $\mathcal{E}_{\rm w}$ is an initial segment of $\mathcal{S}_{\rm w}$. □

We close with some additional remarks.

Remark 4. By relativization, Theorems 1 and 2 also hold with Π^0_1 replaced by boldface $\boldsymbol{\Pi}^0_1$, i.e., *closed*. The lattices of Muchnik degrees of closed subsets of $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ have been studied by Shafer [20].

Remark 5. Shafer [21] suggested that if the density of $\mathcal{E}_{\rm w}$ could be proved, then this might perhaps lead to further progress on the problem of calculating the Turing degree[3] of the first-order theory of $\mathcal{E}_{\rm w}$. Unfortunately, no such further progress has materialized. For further progress, more sophisticated extension-of-embedding results for $\mathcal{E}_{\rm w}$ seem to be needed.

Remark 6. We do not know whether $\mathcal{E}_{\rm w}$ and/or $\mathcal{S}_{\rm w}$ have the *dense splitting* property, i.e., $\forall x \,\forall y\, (x < y \Rightarrow \exists u \,\exists v\, (x < u < y,\, x < v < y,\, \sup(u,v) = y))$. Lachlan [11] proved that $\mathcal{E}_{\rm T}$ does not have the dense splitting property. Binns [4] proved that $\mathcal{E}_{\rm w}$ has the *splitting* property, i.e., the special case $x = 0$. We do not know whether $\mathcal{S}_{\rm w}$ has the splitting property.

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