Contemporary Mathematics

Natural Definability in Degree Structures

Richard A. Shore

Abstract. A major focus of research in computability theory in recent years has involved definability issues in degree structures. There has been much success in getting general results by coding methods that translate first or second order arithmetic into the structures. In this paper we concentrate on the issues of getting definitions of interesting, apparently external, relations on degrees that are order-theoretically natural in the structures $\mathcal{D}$ and $\mathcal{R}$ of all the Turing degrees and of the r.e. Turing degrees, respectively. Of course, we have no formal definition of natural but we offer some guidelines, examples and suggestions for further research.

1. Introduction

A major focus of research in computability theory in recent years has involved definability issues in degree structures. The basic question is, which interesting apparently external relations on degrees can actually be defined in the structures themselves, that is, in the first order language with the single fundamental relation of relative computability, the basic partial ordering $\leq$ on degrees? Most of the work has focused on the Turing degrees and on the structures $\mathcal{R}$ and $\mathcal{D}$ consisting of the recursively enumerable degrees and all the degrees, respectively. We will do the same in this paper.

At the level of establishing abstract definability of relations there has been great success. In both structures, any relation invariant under the double jump whose definability is not ruled out simply by the absolute limitations imposed by the structures being themselves subsystems of first or second order arithmetic, respectively, is actually definable in the structures.

Definition 1.1. *An n-ary relation $P(\mathbf{x}_1, \dots, \mathbf{x}_n)$ on $\mathcal{R}$ is* invariant under the double jump *if, whenever $\mathcal{R} \models P(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $\mathbf{x}_1'' \equiv_T \mathbf{y}_1'', \dots, \mathbf{x}_n'' \equiv_T \mathbf{y}_n''$, it is also true that $\mathcal{R} \models P(\mathbf{y}_1, \dots, \mathbf{y}_n)$. We say that P is* invariant in $\mathcal{R}$ *if whenever $\mathcal{R} \models P(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and φ is an automorphism of $\mathcal{R}$, $\mathcal{R} \models P(\varphi(\mathbf{x}_1), \dots, \varphi(\mathbf{x}_n))$.*

1991 *Mathematics Subject Classification.* Primary 03D25, 03D39; Secondary 03D55, 03E45.

Key words and phrases. degrees, Turing degrees, recursively enumerable degrees, natural definabiltiy.

Partially supported by NSF Grant DMS-9802843. I would also like to thank T. A. Slaman and S. G. Simpson for their helpful comments.

THEOREM 1.2. (Nies, Shore and Slaman [1998]) *Any relation on $\mathcal{R}$ which is invariant under the double jump is definable in $\mathcal{R}$ if and only if it is definable (on indices) in first order arithmetic.*

The route to this result is rather complicated. It begins by coding $\mathbb{N}$, the standard model of arithmetic, in $\mathcal{R}$ in the sense of interpretations of one structure in another as in Hodges [1993]. The coding uses parameters in such a way that, when they are chosen to satisfy some definable condition, the formulas define such structures. One then divides by a definable equivalence relation to get a single model.

THEOREM 1.3. (Nies, Shore and Slaman [1998]) *(i) There is a uniformly definable class $\mathcal{C}_{st}$ of coded standard models of arithmetic.*
(ii) Let $\tilde{\mathbf{N}} = \{(\mathbf{x}, \overline{\mathbf{p}}) : M(\overline{\mathbf{p}}) \in \mathcal{C}_{st} \wedge \mathbf{x} \in M(\overline{\mathbf{p}})\}$. The equivalence relation Q on $\tilde{\mathbf{N}}$ given by

$$(\mathbf{x}, \overline{\mathbf{p}})Q(\mathbf{y}, \overline{\mathbf{q}}) \Leftrightarrow (\exists n \in \omega)[\mathbf{x} = n^{M(\overline{\mathbf{p}})} \wedge \mathbf{y} = n^{M(\overline{\mathbf{q}})}]$$

is definable in $\mathcal{R}$.
(iii) A standard model of arithmetic $\mathbf{N}$ can be defined on the set of equivalence classes $\tilde{\mathbf{N}}/Q$ without parameters.

THEOREM 1.4. (Nies, Shore and Slaman [1998]) *There is a definable map $f : \mathcal{R} \to \mathbf{N}$ such that $(\forall \mathbf{a})[\deg(W^{(2)}_{f(\mathbf{a})}) = \mathbf{a}^{(2)}]$.*

PROOF. (Sketch) To give a first-order definition of f, we have to provide an appropriate definable relation R_f which holds between degrees $\mathbf{a}$ and tuples $(i, \overline{\mathbf{p}})$ representing an equivalence class in $\mathbf{N}$. Note that $\mathbf{a}^{(2)}$ is the least degree $\mathbf{v}$ such that each set in $\Sigma^0_3(A)$ is r.e. in $\mathbf{v}$. (If the last statement holds for $\mathbf{v}$, then $A^{(2)}$ and $\overline{A^{(2)}}$ are r.e. in $\mathbf{v}$.) We argue that $\Sigma^0_3(A) = S(\mathbf{a})$, the class of sets coded in a particular way that depends on $\mathbf{a}$. Thus using a first-order way to obtain, from the degree $\mathbf{a}$, representations of $S(\mathbf{a})$ “inside” $\mathbf{N}$ we can define R_f since finding an index for such a least $\mathbf{v}$ is an arithmetical process. R_f then provides the required map from degrees $\mathbf{a}$ to codes of numbers i in $\mathbf{N}$ such that $\mathbf{a}'' = W''_i$. $\square$

A similar result also holds for $\mathcal{D}$ and is the result of a long line of research.

THEOREM 1.5. (Simpson [1977]; Nerode and Shore [1980,1980a]; Jockusch and Shore [1984]; Slaman and Woodin [2000]; Nies, Shore and Slaman [1998]; Shore and Slaman [2000a]) *Any relation on $\mathcal{D}$ which is invariant under the double jump is definable in $\mathcal{D}$ if and only if it is definable in second order arithmetic.*

PROOF. (Sketch) One can begin by coding arbitrary countable relations by Slaman and Woodin [1986] or directly coding models of arithmetic in lattice initial segments with quantification over sets provided by quantification over ideals via their representation as exact pairs as in Simpson [1977] or Nerode and Shore [1980,1980a]. An analysis of the complexity of sets that can be coded in these models by degrees below a given $\mathbf{x} > \mathbf{0}''$ in terms of the degree of $\mathbf{x}$ itself (or more precisely in terms of $\mathbf{x}''$ or $\mathbf{x}'''$) allows one to define a map between degrees $\mathbf{x}$ and sets of degree $\mathbf{x}''$ coded in such models of arithmetic. Given such a coding, one translates definitions in second order arithmetic into $\mathcal{D}$ by using it together with an interpretation of second order arithmetic in $\mathcal{D}$. $\square$

Whatever further success might be won along these lines they have not provided and will not provide "natural" definitions of degrees or relations on the degrees. This investigation is the provenance of another area of long term interest in the study of $\mathcal{R}$ and $\mathcal{D}$: the relationships between order-theoretic properties of degrees and external properties of other sorts. Of particular interest in $\mathcal{R}$ have been set-theoretic properties described in terms of the lattice of r.e. sets and dynamic properties of the enumerations of the r.e. sets. In both $\mathcal{R}$ and $\mathcal{D}$ we have relations with rates of growth of functions recursive in various degrees and relations with definability considerations in arithmetic as expressed by the jump operator as well as others. In $\mathcal{D}$, a central role has also been played by the jump operator itself, its analogs and their iterations into the transfinite. We will also discuss two notions that play fundamental roles in computability theory but often seem difficult to capture in terms of degree theoretic properties alone: uniformity and recursion.

Of course, we have no formal definition determining which definitions are natural. We can say, with Justice Potter Stewart, that we know the unnatural ones when we see them. The examples given above are prime candidates. One clear offense is that they simply copy the standard definitions in (second order) arithmetic into the degrees by translating all of arithmetic into the language of ($\mathcal{D}$) $\mathcal{R}$. Natural definitions, on the other hand should be more directly expressed in the language of partial orderings and preferably be related to structural or algebraic properties already of interest. The artistic merit of such definitions can also be displayed in the notions and constructions used to establish them. They may also have redeeming social value in that the proofs reveal or exploit specific properties of the degrees in the class being defined, for example, in terms of rates of growth, dynamic properties of enumerations or of the external relation being characterized. We discuss some examples and suggest questions for further research. As might be expected, borderline cases will arise. In the end, the naturalness of proposed definitions along with the beauty of the constructions and ideas needed will be judged by the community of readers and researchers in the field.

2. The recursively enumerable degrees

The jump operator itself is, of course, not defined in $\mathcal{R}$ but there are many important results connecting the jump classes of r.e. degrees with the lattice theoretic structure of the r.e. sets; with approximation procedures for functions recursive in different jumps of the given set; and with the growth rates of functions recursive in the sets themselves for several of these jump classes.

Definition 2.1. *An r.e. degree* $\mathbf{a}$ *is* high_n *iff* $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$ *(its* n^{th} *jump is as high as possible). The degree* $\mathbf{a}$ *is* low_n *if* $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ *(its* n^{th} *jump is as low as possible). If* $n = 1$*, we usually omit the subscript.*

Example 2.2. An r.e. degree $\mathbf{a}$ is high iff it contains a maximal set in $\mathcal{E}^*$ (the lattice of r.e. sets modulo the ideal of finite sets) iff there is a function f of degree a which dominates every recursive function iff every function h recursive in $0''$ is approximable by one $g \leq \mathbf{a}$ in the sense that $h(x) = \lim_s g(x, s)$.

Example 2.3. An r.e. degree $\mathbf{a}$ is low_2 iff every r.e. set of degree $\mathbf{a}$ has a maximal superset iff there is a function f recursive in $0'$ which dominates every $g \leq \mathbf{a}$ iff every function $h \leq \mathbf{a}''$ is approximable by a function recursive in g in the sense that $h(x) = \lim_s \lim_t g(x, s, t)$.

(Proofs of these facts about high and low$_2$ r.e. sets and other similar ones can be found in Soare [1987, XI].)

These interrelations have played an important role in the study of both the lattice of r.e. sets (see Soare [1999]) and the degrees below $0'$ (see Cooper [1999]). The results provide definitions in $\mathcal{R}'$ ($\mathcal{R}$ with an added predicate for two degrees having the same jump) but no natural degree theoretic definitions within $\mathcal{R}$ itself.

QUESTION 2.4. *Are any of the jump classes $\mathbf{H}_n, \mathbf{L}_{n+1}$ naturally definable in $\mathcal{R}$? In particular, what about $\mathbf{L}_2$ and $\mathbf{H}_1$ where we have the strongest connections with rates of growth and significant techniques already developed to exploit the known characterizations of these classes?*

If we omit the requirement of naturalness then this question was raised in Soare [1987] and answered affirmatively for $n \geq 1$ by Nies, Shore and Slaman [1998] with Theorem 1.2 and Proposition 4.3. The first natural, truly internal definitions in $\mathcal{R}$ of apparently external properties of r.e. degrees arose from the study of Maass' [1982] notion of prompt simplicity.

DEFINITION 2.5. *A coinfinite r.e. set A is* promptly simple *if there is there is a nondecreasing recursive function p and a recursive one-to-one function f enumerating A (i.e. $A = \operatorname{rg} f$) such that for every infinite r.e. W_e there is an s and an x such that x is enumerated in W at stage s (in some standard uniform enumeration of all the r.e. sets) and is also enumerated in A by stage $p(s)$, i.e. $x = f(n)$ for some $n \leq p(s)$. An r.e. degree $\mathbf{a}$ is* promptly simple *if it contains a promptly simple r.e. set. We let $\mathbf{PS}$ denote the set of promptly simple r.e. degrees.*

DEFINITION 2.6. *i) $\mathbf{M} = \{\mathbf{a} | \exists \mathbf{b}(\mathbf{a} \wedge \mathbf{b} = \mathbf{0})\}$ is the set of* cappable *r.e. degrees, i.e. those which are halves of minimal pairs. $\mathbf{NC}$, the set of* noncappable *r.e. degrees, is its complement in $\mathcal{R}$.*
ii) $\mathbf{LC} = \{\mathbf{a} | \exists \mathbf{b}(\mathbf{a} \vee \mathbf{b} = \mathbf{0}' \,\&\, \mathbf{b}' = \mathbf{0}')\}$ is the set of low cuppable degrees*, i.e. those which can be cupped (joined) to $\mathbf{0}'$ by a low degree $\mathbf{b}$.*
iii) $\mathbf{SP\bar{H}}$ is the set of r.e. sets definable in $\mathcal{E}$, the lattice of r.e. sets, as the non-hyperhypersimple r.e. sets A with the splitting property*, i.e. for every r.e. set B there are r.e. sets B_0, B_1 such that $B_0 \cup B_1 = B$; $B_0 \cap B_1 = \emptyset$; $B_0 \subseteq A$; and if W is r.e. but $W - B$ is not, then $W - B_0$ and $W - B_1$ are also not r.e.*

THEOREM 2.7. (Ambos-Spies et al. [1984]) *The four classes $\mathbf{PS}, \mathbf{NC}, \mathbf{LC}$ and $\mathbf{SP\bar{H}}$ all coincide and together with their complement $\mathbf{M}$ partition $\mathcal{R}$ as follows:*

i) $\mathbf{M}$ is a proper ideal in $\mathcal{R}$, i.e. it is closed downward in $\mathcal{R}$ and if $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ then $\mathbf{a} \vee \mathbf{b} \in \mathbf{M}$ and $\mathbf{a} \vee \mathbf{b} < \mathbf{0}'$.

ii) $\mathbf{NC}$ is a strong filter in $\mathcal{R}$, i.e. it is closed upward in $\mathcal{R}$ and if $\mathbf{a}, \mathbf{b} \in \mathbf{NC}$ then there is a $\mathbf{c} \in \mathbf{NC}$ with $\mathbf{c} \leq \mathbf{a}, \mathbf{b}$.

We would also like to point out two hidden uniformities in these results. Their proofs provide a recursive function f such that if $\deg(W_e) \in \mathbf{M}$ then W_e and $W_{f(e)}$ form a minimal pair. They also shows that $\mathbf{NC} = \mathbf{ENC}$, the effectively noncappable degrees, i.e. those $\mathbf{a}$ such that there is an r.e. $A \in \mathbf{a}$ and a recursive function f such that for all e, $W_e \leq_T A$ uniformly in e; $W_{f(e)} \leq_T W_e$ uniformly in e; and if W_e is not recursive then $W_{f(e)}$ is not recursive either.

QUESTION 2.8. *As Slaman has pointed out, the noncuppable degrees (those degrees $\mathbf{a}$ for which there is no $\mathbf{b} < \mathbf{0}'$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$) trivially also form an ideal in $\mathcal{R}$. Are there any other (naturally) definable ideals or filters in $\mathcal{R}$?*

QUESTION 2.9. *If $\mathcal{B}$ is a (particular) definable subset of $\mathcal{R}$ is there a way to define the ideal generated by $\mathcal{B}$?*

The only other known example of a natural definition in $\mathcal{R}$ is that of contiguity, a notion relating Turing reducibility and weak truth-table reducibility, $\leq_{wtt}$. As wtt-reducibility is the restriction of Turing reducibility to operators with a recursive bound on their use functions, this characterization provides an example of defining a notion apparently dependent on external computational procedures solely in terms of Truing reducibility. Indeed, in this case we have two equivalent definitions in $\mathcal{R}$ with the second being a later refinement of the first.

DEFINITION 2.10. *An r.e. degree $\mathbf{a}$ is* contiguous *if and only if for every r.e. $A, B \in \mathbf{a}$, $A \equiv_{wtt} B$. It is* strongly contiguous *if and only if for every $A, B \in \mathbf{a}$, $A \equiv_{wtt} B$ (i.e. $\mathbf{a}$ consists of a single wtt-degree).*

THEOREM 2.11. (Downey and Lempp [1997]) *An r.e. degree $\mathbf{a}$ is contiguous if and only if it is strongly contiguous if and only if it is* locally distributive*, i.e. if $\mathbf{b} \vee \mathbf{c} = \mathbf{a} > \mathbf{d}$ then $\exists \mathbf{b}_0 \leq \mathbf{b} \exists \mathbf{c}_0 \leq \mathbf{c}(\mathbf{b}_0 \vee \mathbf{c}_0 = \mathbf{d})$.*

THEOREM 2.12. (Ambos-Spies and Fejer [2000]) *An r.e. degree $\mathbf{a}$ is contiguous if and only if it is not the top of an embedding of the pentagon into $\mathcal{R}$ (if and only if there is an r.e. $A \in \mathbf{a}$ with the strong universal splitting property).*

An interesting property that seems to capture the ability to do multiple permitting in a way intermediate between the permitting arguments that just use nonrecursiveness and those that use nonlow$_2$-ness is that of being array nonrecursive.

DEFINITION 2.13. (Downey, Jockusch and Stob [1990]) *An r.e. degree $\mathbf{a}$ is* array nonrecursive *iff there is an r.e. $A \in \mathbf{a}$ and a very strong array F_n such that $\forall e \exists n (W_e \cap F_n = A \cap F_n)$ or equivalently there are disjoint r.e. $B, C \leq_T A$ such that $B \cup C$ is coinfinite and no $D \in \mathbf{0}'$ separates B and C. (The sequence F_n is a* very strong array *if there is a recursive function f such that $f(n)$ is the canonical index for F_n, $\bigcup F_n = \mathbb{N}$, the F_n are pairwise disjoint and of strictly increasing cardinality.)*

QUESTION 2.14. (Walk)*: Are the array recursive degrees (naturally) definable in $\mathcal{R}$?*

In fact, Walk had suggested that the array nonrecursive degrees might be those bounded by a contiguous degree. This turns out to be false but the work needed to prove this provides a definable automorphism base for $\mathcal{R}$, i.e. a set $\mathbf{MC}$ of degrees such that any automorphism of $\mathcal{R}$ which is fixed on $\mathbf{MC}$ is the identity on $\mathcal{R}$.

THEOREM 2.15. (Cholak, Downey and Walk [2000]) *There is a maximal contiguous degree and indeed the set, $\mathbf{MC}$, of maximal contiguous degrees forms an automorphism base for $\mathcal{R}$.*

We close this section with a question about relativizations. It was a long standing question (originally asked in Sacks [1963a]) as to whether $\mathcal{R}^{\mathbf{a}}$, the degrees r.e. in and above $\mathbf{a}$, are isomorphic (or elementary equivalent to) $\mathcal{R}^{\mathbf{b}}$, those r.e. in and above $\mathbf{b}$, for arbitrary $\mathbf{a}$ and $\mathbf{b}$. The question about isomorphisms was answered negatively in Shore [1982] and for elementary equivalence in Nies, Shore and Slaman [1998]. The first result distinguishes them on the basis of which infinite lattices (of degrees related to $\mathbf{a}$ and $\mathbf{b}$) can be embedded and the second on the basis of which sets (again of degrees related to $\mathbf{a}$ and $\mathbf{b}$) can be definably coded in models of arithmetic represented in the structures. Of course, neither of these approaches provides a natural difference between the structures.

QUESTION 2.16. (Simpson) *Is there a natural sentence ϕ of degree theory that is true in some $\mathcal{R}^{\mathbf{a}}$ but not in all $\mathcal{R}^{\mathbf{b}}$? If so, for which $\mathbf{a}$ and $\mathbf{b}$ can one find such sentences?*

3. The Turing degrees

We begin our discussion of natural definability in $\mathcal{D}$ with the first example of a natural definition of an interesting, seemingly external, class of degrees, those of the arithmetic sets.

DEFINITION 3.1. *A degree $\mathbf{a}$ is a* minimal cover *if there is a $\mathbf{b}$ such that $\mathbf{a} > \mathbf{b}$ and there is no $\mathbf{c}$ strictly between $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{b}$ is such a degree for $\mathbf{a}$ we say that $\mathbf{a}$ is a* minimal cover of $\mathbf{b}$.

DEFINITION 3.2. $\mathcal{A} = \{\mathbf{d} | \exists n(\mathbf{d} \leq \mathbf{0}^{(n)})\}$. $\mathcal{C}_\omega = \{\mathbf{c} | \forall \mathbf{z}(\mathbf{z} \vee \mathbf{c}$ *is not a minimal cover of* $\mathbf{z}\}$. $\overline{\mathcal{C}}_\omega = \{\mathbf{d} | \exists \mathbf{c} \in \mathcal{C}_\omega(\mathbf{d} \leq \mathbf{c})\}$.

THEOREM 3.3. (Jockusch and Shore [1984]) $\mathcal{A} = \overline{\mathcal{C}}_\omega$ *and the relation $\mathbf{a}$ is arithmetic in $\mathbf{b}$ is naturally definable in $\mathcal{D}$ (by relativization).*

This result was proved by combining a completeness and join theorem for certain operators with known structural results for $\mathcal{D}$ and the r.e. degrees. We begin with the definitions of the operators studied. (We restrict ourselves here to the cases that $\alpha \leq \omega$ but both of the following definitions have been usefully generalized into the transfinite and the corresponding theorems proven.)

DEFINITION 3.4. *The* $1 - REA$ operators J *(from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$) are those of the form $J(A) = J_e(A) = A \oplus W_e^A$. The* $n - REA$ operators J *are those of the form $J_{\langle e_1, \ldots, e_n \rangle} = J_{e_n} \circ J_{e_{n-1}} \circ \ldots \circ J_{e_1}$. The* $\omega - REA$ operators J *are those of the form $J(A) = \bigoplus\{J_{f \restriction n}(A) | n \in \omega\}$ for some recursive f.*

DEFINITION 3.5. *The* $n - r.e.$ operators J *are those of the form $J(A)(x) = \lim \phi_e^A(x, s)$ for a (total recursive in A) function ϕ_e^A which has $\phi_e^A(x, 0) = 0$ for all x and for which there are at most n many s such that $\phi_e^A(x, s) \neq \phi_e^A(x, s+1)$. The* $\omega - r.e.$ operators J *are those of the form $J(A)(x) = \lim \phi_e^A(x, s)$ for a (total recursive in A) function ϕ_e^A which has $\phi_e^A(x, 0) = 0$ for all x and for which there are at most $f(x)$ many s such that $\phi_e^A(x, s) \neq \phi_e^A(x, s+1)$ for some recursive function f.*

Now for the completeness theorem generalizing those of Friedberg [1957] and MacIntyre [1977].

THEOREM 3.6. (Jockusch and Shore [1984]) *For any $\alpha - REA$ operator J and any $C \geq_T 0^{(\alpha)}$ there is an A such that $J(A) \equiv_T C$.*

Applying this theorem to specific operators provides suggestions for producing interesting definable subsets of $\mathcal{D}$. We consider the minimal degree construction but clearly others are possible.

THEOREM 3.7. (Sacks [1963]) *There is an $\omega - r.e.$ operator J such that $J(A)$ is a minimal cover of A for every A.*

COROLLARY 3.8. $\mathbf{0}^{(\omega)}$ *is the base of a cone of minimal covers, i.e. every* $\mathbf{a} \geq \mathbf{0}^{(\omega)}$ *is a minimal cover.*

QUESTION 3.9. *Is* $\mathbf{0}^{(\omega)}$ *the least degree which is the base of a cone of minimal covers?*

We conjecture that the answer to this question is no and suggest that one build a tree T such that every path through T is a minimal cover by virtue of being $J(A)$ for some A while simultaneously making the degree of T incomparable with $\mathbf{0}^{(\omega)}$. If this is possible, we ask instead what can one say about the degrees which are bases of cones of minimal covers?

An improvement to the completeness theorem that includes a join operation provides an approach to our first natural definability result for $\mathcal{D}$.

THEOREM 3.10. (essentially as in Jockusch and Shore [1984]) *For any* $\alpha - r.e.$ *operator* J *(* $1 \leq \alpha \leq \omega$*) and any* $D \not\leq_T 0^{(\beta)}$ *for every* $\beta < \alpha$ *there is an* A *such that* $J(A) \equiv_T D \vee 0^{(\alpha)} \equiv_T A^{(\alpha)}$.

Thus for every nonarithmetic degree $\mathbf{x}$ there is a $\mathbf{z}$ such that $\mathbf{x} \vee \mathbf{z}$ is a minimal cover of $\mathbf{z}$. On the other hand, $\mathbf{0}^{(n)} \vee \mathbf{z}$ is not a minimal cover of $\mathbf{z}$ for any $\mathbf{z}$ by Jockusch and Soare [1970]. This establishes Theorem 3.3.

Cooper [1990, 1993, 1994 and elsewhere] suggested a similar approach to the problem of defining the jump operator. His plan was to use Theorem 3.10 to define $\mathbf{0}'$ by finding a suitable $2 - r.e.$ operator (rather than an $\omega - r.e.$ one as above) that would produce a degree with an order-theoretic property that no r.e. degree could have (again even relative to any degree below it). He defined the following notions and classes.

DEFINITION 3.11. $\mathbf{d}$ *is* splittable over $\mathbf{a}$ avoiding $\mathbf{b}$ *if either* $\mathbf{a}, \mathbf{b} \not\leq \mathbf{d}$ *or* $\mathbf{b} \leq \mathbf{a}$ *or there are* $\mathbf{d}_0, \mathbf{d}_1$ *such that* $\mathbf{a} < \mathbf{d}_0, \mathbf{d}_1 < \mathbf{d}$*,* $\mathbf{d}_0 \vee \mathbf{d}_1 = \mathbf{d}$ *and* $\mathbf{b} \not\leq \mathbf{d}_0, \mathbf{d}_1$. $\mathcal{C}_1 = \{\mathbf{c} | \forall \mathbf{a}, \mathbf{b} (\mathbf{a} \vee \mathbf{c}$ *is splittable over* $\mathbf{a}$ *avoiding* $\mathbf{b}\}$. $\overline{\mathcal{C}}_1 = \{\mathbf{d} | \exists \mathbf{c} \in \mathcal{C}_1 (\mathbf{d} \leq \mathbf{c})\}$.

Now, one of the needed results was already well known.

THEOREM 3.12. (Sacks [1963]) *Every r.e. degree* $\mathbf{d}$ *is in* $\mathcal{C}_1$.

For the other direction Cooper [1990, 1993] claimed as his main theorem that there is a suitable $2 - r.e.$ set and so $2 - r.e.$ operator J such that for every C there are $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{d} \equiv_T \deg(J(C))$ is not splittable over $\mathbf{a}$ avoiding $\mathbf{b}$.

Such a result would provide a natural definition of $\mathbf{0}'$ as the maximum degree in $\overline{\mathcal{C}}_1$ and, by relativization, a natural definition of the jump operator. Unfortunately, there is no such $2 - r.e.$ operator. Nor, indeed, any $n - REA$ one.

THEOREM 3.13. (Shore and Slaman [2000a]) *If* $\mathbf{a}, \mathbf{b} \leq_{\mathbf{T}} \mathbf{d}$*,* $\mathbf{b} \not\leq_{\mathbf{T}} \mathbf{a}$ *and* $\mathbf{d}$ *is* $n - REA$ *in* $\mathbf{a}$*, then* $\mathbf{d}$ *can be split over* $\mathbf{a}$ *avoiding* $\mathbf{b}$.

A number of other specific suggestions of natural properties that might define $\mathbf{0}'$ have been made and most have been refuted. Posner [1980] conjectured that $\mathbf{0}'$ is the least degree $\mathbf{x}$ such that every $\mathbf{d} \geq \mathbf{x}$ is the join of two minimal degrees or [1981] such that $\mathcal{D}(\leq \mathbf{d})$ is complemented for every $\mathbf{d} \geq \mathbf{x}$. One can refute the first conjecture by constructing a function $f : \omega \to \{0, 1, 2\}$ such that $f |_T 0'$ and $(\forall g : \omega \to \{0, 1\})(\forall x [f(x) \in \{0, 1\} \to f(x) = g(x)] \to \deg(g)$ is minimal). Given any $h \geq_T f$ one defines g_0, g_1 as in the construction of f such that $g(a_i) = h(i)$ and $g_1(a_i) = 1 - h(i)$ where $\langle a_i : i \in \omega \rangle$ lists the numbers x such that $f(x) = 2$. Thus

$h \equiv_T g_1 \oplus g_2$. The construction of f follows Lachlan [1971] with added steps to make f incomparable with $0'$. Slaman and Steel [1989] ask if $\mathbf{0}'$ is the least $\mathbf{x}$ such that for every $\mathbf{d} \geq \mathbf{x}$ there is an $\mathbf{a}$ such that every nonzero $\mathbf{b} \leq \mathbf{a}$ is a complement for every $\mathbf{c} \geq \mathbf{x}$ in $\mathcal{D}(\leq \mathbf{d})$ but now expect that this proposal will also fail.

At the Boulder meeting we proposed that direct natural definitions of the $\mathbf{0}^{(n)}$ might be proved by strengthening Theorem 3.10 to handle all $n - REA$ operators and then finding an appropriate property of some $n - REA$ degrees along the lines suggested by Cooper for $2-r.e.$ Thus one should first answer the following question:

QUESTION 3.14. (Jockusch and Shore [1984]) *For each $\alpha > 1$, is there, for each $\alpha - REA$ operator J ($\alpha > 1$) and each D such that $D \not\leq_T \mathbf{0}^{(\beta)}$ for every $\beta < \alpha$, an A such that $J(A) \equiv_T D \vee A \equiv_T D \vee 0^{(\alpha)}$.*

For $\alpha = 1$, the result follows from Posner and Robinson [1981] for which a new proof is supplied in Jockusch and Shore [1984] as the paradigm for that of Theorem 3.10. The case of $\alpha = \omega$ is singled out in Jockusch and Shore [1985] because of the role it played in the proof of Theorem 3.3. This case was solved by Kumabe and Slaman (personal communication). During the Boulder conference, Shore and Slaman discussed refining the construction to apply to $\alpha < \omega$ (a more delicate question as it turns out). They have now answered that question.

THEOREM 3.15. (Shore and Slaman [2000]) *For each $(n+1) - REA$ operator J ($n \geq 0$) and $D \not\leq_T 0^{(n)}$ there is an A such that $J(A) \equiv_T D \vee A = D \vee 0^{(n+1)}$.*

Thus, if one can find a (natural) property of some $(n+1) - REA$ degree that is not enjoyed by any $n - REA$ degree and both of these facts relativize, then one would have a (natural) direct definition of $\mathbf{0}^{(n)}$. Indeed, it was in a attempt to produce such a definition of $\mathbf{0}'$ by trying to construct a $2 - REA$ degree $\mathbf{d}$ and $\mathbf{a}, \mathbf{b} \leq \mathbf{d}$ such that $\mathbf{d}$ is not splittable over $\mathbf{a}$ avoiding $\mathbf{b}$ that Shore and Slaman were led to Theorem 3.13. They then realized that there was a property that Slaman and Woodin had shown to be definable that would play the required role, that of being a double jump.

THEOREM 3.16. (Slaman and Woodin [2000]) $\mathbf{0}''$ *and, indeed, the operator taking* $\mathbf{a}$ *to* $\mathbf{a}''$ *is definable in $\mathcal{D}$.*

Slaman and Woodin had discovered this result in 1990 but never published or even publicized it as Cooper had already announced that $\mathbf{0}'$ and the jump operator itself were definable in $\mathcal{D}$. It does, however, supply the missing ingredient for a definition of the jump.

THEOREM 3.17. (Shore and Slaman [2000]) $\mathbf{0}'$ *and the jump operator are definable in $\mathcal{D}$.*

PROOF. The claim is that $\mathbf{0}'$ is the greatest degree $\mathbf{x}$ such that there is no $\mathbf{b}$ for which $\mathbf{b}'' = \mathbf{x} \vee \mathbf{b}$. In one direction, it is clear that if $\mathbf{x} \leq \mathbf{0}'$ then $\mathbf{x} \vee \mathbf{b} \leq \mathbf{b}' < \mathbf{b}''$ for every $\mathbf{b}$ and so $\mathbf{x}$ has the desired property. For the other direction, if $\mathbf{x} \not\leq \mathbf{0}'$ then by Theorem 3.10 there is a $\mathbf{b}$ such that $\mathbf{b}'' = \mathbf{x} \vee \mathbf{b}$. Finally, $\mathbf{b}''$ is definable (from $\mathbf{b}$) by Theorem 3.16 while, of course, the join is definable from $\leq_T$. This then defines the degree $\mathbf{0}'$. The jump operator (applied to any $\mathbf{z}$) is defined by relativizing the theorems and definitions to $\mathbf{z}$. $\square$

Interestingly, the proof of Theorem 3.16 proceeds by applying metamathematical arguments involving forcing and absoluteness. Concomitantly, the definition of $\mathbf{0}''$ that these arguments provide involves an explicit translation of isomorphism facts to definability facts via a coding of (second order) arithmetic. Thus, unfortunately, we must say that the definition provided is not natural. We do, however, expect that a more directly expressible appropriate property of $2-REA$ degrees will be found that will supply the desired natural definition. Indeed, Cooper has announced that a modification of his original $2-r.e.$ operator does provide such a definition of the jump and of the relation "$\mathbf{a}$ is r.e. in $\mathbf{b}$". In any case, we would like to see such direct definitions for all the $\mathbf{0}^{(n)}$.

QUESTION 3.18. *For each n what properties are enjoyed by some $(n+1)-REA$ or even $(n+1)-r.e.$ degree which are not enjoyed by any $n-REA$ degree? (Note that by Jockusch and Shore [1984] every $n-r.e.$ degree is $n-REA$.)*

This problem suggests a general area of investigation.

QUESTION 3.19. *What special properties do the $n-REA$ or $n-r.e.$ degrees have within the whole structure $\mathcal{D}$?*

QUESTION 3.20. *Is the class of $n-REA$ degrees and the relation "$\mathbf{a}$ is $n-REA$ in $\mathbf{b}$" (naturally) definable for each n?*

Now if one has a natural definition of the relation "$\mathbf{a}$ is r.e. in $\mathbf{b}$" then, of course, one can define these relations by iterating the assumed definition for $n=1$. It then becomes debatable at what point (if any) such iterations cease to provide natural definitions. In any case, it would be of interest to provide direct definitions for each n that directly exploit some aspect of being $n-REA$.

The next question might well be about the union of these classes: $\mathbf{REA}^{<\omega} = \{\mathbf{a} | \exists n \in \omega(\mathbf{a} \text{ is } n-REA)\}$. It is not, perhaps, immediately clear, even assuming the definability of "$\mathbf{a}$ is r.e. in $\mathbf{b}$", that $\mathbf{REA}^{<\omega}$ is definable in $\mathcal{D}$ let alone naturally. How do we quantify over $n \in \omega$? Well, it would clearly suffice to quantify over finite sequences (assuming the definability of "$\mathbf{a}$ is r.e. in $\mathbf{b}$"). This we can certainly do by Slaman-Woodin [1986] coding. Should this count as a natural definition? It is phrased in order-theoretic terms but is perilously close to defining arithmetic. Perhaps some more traditional methods can be employed. A similar question arises for defining the set $\{\mathbf{0}^{(n)} | n \in \omega\}$ from a definition of the jump operator. In this case Simpson has supplied an answer.

THEOREM 3.21. (Simpson [1977]) *If $\langle \mathbf{b}_n | n \in \omega \rangle$ is a sequence of degrees such that $\mathbf{0}'' \leq \mathbf{b}_1 \leq \ldots \leq \mathbf{b}_n \leq \ldots$, then there an initial segment $0 < \mathbf{a}_1 < \ldots < \mathbf{a}_n < \ldots$ of $\mathcal{D}$ such that $\mathbf{a}_i'' = \mathbf{a}_i \vee \mathbf{0}'' = \mathbf{b}_i$ for $i \geq 1$.*

COROLLARY 3.22. $\{\mathbf{0}^{(n)} | n \in \omega\} = \{\mathbf{d} | \mathbf{d} = \mathbf{0} \vee \mathbf{d} = \mathbf{0}' \vee$ *there is a finite initial segment of the degrees with top degree $\mathbf{a}$ which is linearly ordered and such that the double jump of each non zero element is the jump its immediate predecessor and $\mathbf{d}$ is the double jump of some $\mathbf{c} < \mathbf{a}$*$\}$.

The only part of this definition that is not obviously in the language of $\mathcal{D}$ with jump is the assertion that the linearly ordered initial segment with top $\mathbf{a}$ is *finite*. The crucial ingredient here is Spector's exact pair theorem that we shall see mercilessly exploited below (as in Jockusch and Simpson [1976]).

DEFINITION 3.23. *An* ideal *of $\mathcal{D}$ is a subset I of $\mathcal{D}$ which is closed downward and under join. It is a* jump ideal *if it is also closed under jump. We say that a pair $\mathbf{x}, \mathbf{y}$ of degrees is an* exact pair *for an ideal I if $I = \{\mathbf{z} | \mathbf{z} \leq \mathbf{x}$ and $\mathbf{z} \leq \mathbf{y}\}$.*

THEOREM 3.24. (Spector [1956]) *Every countable ideal I of $\mathcal{D}$ has an exact pair.*

Using this theorem one can refer to countable ideals or quantify over them in a first order way by using the corresponding exact pairs. Given this it is easy to say that a linear ordering is finite by saying that every countable initial segment has a greatest element.

A similar line of argument provides a natural definition of the intermediate r.e. degrees (from the relation of being r.e. in).

COROLLARY 3.25. *The set of intermediate r.e. degrees, i.e.* $\{\mathbf{x}|\mathbf{x}$ *is r.e. and* $\forall n(\mathbf{0}^{(n)} < \mathbf{x}^{(n)} < \mathbf{0}^{(n+1)})\}$, *is definable as* $\{\mathbf{x}|\mathbf{x}$ *is r.e. and for every finite initial segment of the degrees with top degree* $\mathbf{a}$ *which is linearly ordered and such that the double jump of each nonzero element is the jump of its immediate predecessor there is another finite initial segment of the degrees such that the double jump of the first nonzero element is* $\mathbf{x}''$ *and the double jump of each later element is the jump of its immediate predecessor and the double jumps of the second sequence are interleaved with those of the first*$\}$.

Can we get a similar natural definition for $\mathbf{REA}^{<\omega}$? One route is suggested by the following question.

QUESTION 3.26. *If* $\mathbf{a}$ *is* $n-REA$ *as witnessed by the sequence* $\mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_n = \mathbf{a}$, *is there an initial segment* $\mathbf{0} < \mathbf{b}_1 < \ldots < \mathbf{b}_n$ *and a* $\mathbf{c}$ *such that* $\mathbf{a}_i = \mathbf{b}_i \vee \mathbf{c}$*?*

This analysis suggests a general area for investigation that asks to what extent one can control the jump or double jump of all degrees in some initial segment of $\mathcal{D}$.

QUESTION 3.27. *What is the range of the jump or double jump operator on initial segments, i.e. for which sets* C *of degrees above* $\mathbf{0}'(\mathbf{0}'')$ *can we find an initial segment* I *of* $\mathcal{D}$ *such that the range of the jump operator on* I *and/or* $I \vee \mathbf{0}'$ $(I \vee \mathbf{0}'')$ *is* C*? Or prove that some interesting classes of degrees are so representable.*

QUESTION 3.28. *Which sets of degrees are of the form* $I \vee \mathbf{c}$ *for an initial segment* I *of* $\mathcal{D}$ *and a degree* $\mathbf{c}$*? Or prove that some interesting classes of degrees are so representable.*

Actually, in the case of $n-REA$ we might be able to do even better and supply a natural definition that does not appear to mention recursion even indirectly.

PROPOSITION 3.29. (Jockusch and Shore [1984]) $\mathcal{C}_\omega$ *is closed under the relation* $n - REA$ *in, i.e. if* $\mathbf{a} \in \mathcal{C}_\omega$ *and* $\mathbf{b}$ *is* $n - REA$ *in* $\mathbf{a}$ *then* $\mathbf{b} \in \mathcal{C}_\omega$.

QUESTION 3.30. *Does* $\mathcal{C}_\omega = \mathbf{REA}^{<\omega}$*?*

The questions presented here about $\{\mathbf{0}^{(n)}|n \in \omega\}$ and $\mathbf{REA}^{<\omega}$ suggest another array of related issue involving uniformity and upper bounds as represented by the problems of defining $\mathbf{0}^{(\omega)}$ and the ω-REA degrees.

DEFINITION 3.31. *If* S *is a set of functions then a degree* $\mathbf{a}$ *is a* uniform upper bound (uub) *for* S *if there is a function* $f \leq_T \mathbf{a}$ *such that* $\{f^{[i]}|i \in \omega\} = S$ *where* $f^{[i]}(x) = f(\langle i, x\rangle)$. *It is a* subuniform upper bound (suub) *for* S *if there is a function* $f \leq_T \mathbf{a}$ *such that* $\{f^{[i]}|i \in \omega\} \supseteq S$.

The classic example here is $0^{(\omega)}$ which is a uub for the $0^{(n)}$. There is a long history of attempts to characterize or define in $\mathcal{D}$ (or $\mathcal{D}$ with the jump operator) the degrees of uub's, suub's and other types of upper bounds for the recursive and arithmetic sets and functions as well as other (jump) ideals of $\mathcal{D}$. We mention a couple of results and questions.

THEOREM 3.32. (Jockusch [1972]) *A degree* $\mathbf{a}$ *is the degree of a (sub)uniform upper bound of the recursive functions iff it is a uub for the recursive sets iff* $\mathbf{a}' \geq \mathbf{0}''$.

THEOREM 3.33. (Jockusch [1972]; Jockusch and Simpson [1976]; Lachlan and Soare [1994]; Lerman[1985]) *A degree* $\mathbf{a}$ *is the degree of a uniform upper bound of the arithmetic functions iff there is an* $f \in \mathbf{a}$ *which dominates every partial (total) arithmetic function iff there is a* $\mathbf{d}$ *which is an upper bound for the arithmetic degrees and* $\mathbf{a} = \mathbf{d}'$ *iff there is a* $\mathbf{d}$ *which is an upper bound for the arithmetic degrees and* $\mathbf{a}' = \mathbf{d}''$.

QUESTION 3.34. *Are there natural definitions of the property of being the degree of a uniform upper bound for other specific (jump) ideals, of being the degree of a subuniform upperbound for* $\mathcal{A}$ *or other (jump) ideals?*

The hierarchy generated by iterating the jump into the transfinite and taking some kind of uniform upper bound at limit levels has been an object of intense study since Kleene and Spector. Major contributions were also made by Putnam and his students culminating in Hodes [1982]. Jockusch and Simpson [1976] is a remarkable collection of natural definitions (using the jump operator) of many specific natural upper bounds for jump ideals as well as the ideals themselves. They also tie quantification over the functions in such ideals to standard subsystems of second order arithmetic in a definable way. We mention a few of the results.

THEOREM 3.35. (Sacks [1971]; Enderton and Putnam [1970]) $\mathbf{0}^{(\omega)}$ *is the largest degree below* $(\mathbf{a} \vee \mathbf{b})^{(2)}$ *for every exact pair* $\mathbf{a}, \mathbf{b}$ *for* $\mathcal{A}$. *It is also the least degree of the form* $(\mathbf{a} \vee \mathbf{b})^{(2)}$ *for any exact pair* $\mathbf{a}, \mathbf{b}$ *for* $\mathcal{A}$.

Of course, given the jump operator we may also say that an ideal (as given by an exact pair) is a jump ideal. The following theorem is the key to defining many interesting properties of jump ideals in $\mathcal{D}$.

THEOREM 3.36. (Jockusch and Simpson [1976]) X *is* Δ^1_n *over the set* M_I *of functions with degrees in a countable jump ideal* I *if and only if* $\mathbf{h} \leq (\mathbf{a} \vee \mathbf{b})^{(n+1)}$ *for every exact pair* $\mathbf{a}, \mathbf{b}$ *for* I. X *is analytical (i.e.* Σ^1_k *for some* k*) over this set of functions if and only if* $\mathbf{h}$ *is arithmetic in* $\mathbf{a} \vee \mathbf{b}$ *for every exact pair* $\mathbf{a}, \mathbf{b}$ *for* I.

COROLLARY 3.37. (Jockusch and Simpson [1976]) *The property that* M_I *is (a collection of functions corresponding to) an* ω*-model of* Δ^1_n*-Comprehension (*Δ^1_n*-CA) is naturally definable in* $\mathcal{D}$ *by "every* $\mathbf{d}$ *below* $(\mathbf{a} \vee \mathbf{b})^{(n+1)}$ *for every exact pair for* I *is itself in* I*".*

COROLLARY 3.38. (Jockusch and Simpson [1976]) *The jump ideals* $\mathcal{A}$ *and* $\mathcal{H}$ *(the hyperarithmetic degrees) and the relations "*$\mathbf{a}$ *is arithmetic in* $\mathbf{b}$*" and "*$\mathbf{a}$ *is hyperarithmetic (i.e.* Δ^1_1*) in* $\mathbf{b}$*" are naturally definable in* $\mathcal{D}$ *with the jump operator. (For example,* $\mathbf{a}$ *is arithmetic in* $\mathbf{b}$ *iff it is in every countable jump ideal containing* $\mathbf{b}$ *while it is hyperarithmetic in* $\mathbf{b}$ *iff it is every countable jump ideal containing* $\mathbf{b}$ *which is a model of* Δ^1_1*-CA.)*

We have already seen (Theorem 3.3) a definition of $\mathcal{A}$ in $\mathcal{D}$ that does not explicitly refer to the jump operator. It can also be used to produce an alternative definition of $\mathcal{H}$ as well.

PROPOSITION 3.39. (Jockusch and Shore [1984]) *$\mathbf{a}$ is hyperarithmetic in $\mathbf{b}$ iff it is in every countable ideal I containing $\mathbf{b}$ such that every degree $\mathbf{d}$ which is arithmetic in every upper bound for I is actually in I.*

The interest in alternate definitions of concepts is, we would say, a positive indicator of the naturalness of the definitions. Providing new coding methods for producing definitions can be of technical interest but rarely sheds any additional light on the classes or degrees being defined. In contrast, new natural definitions typically exploit different properties of $\mathcal{D}$ or of the objects being defined and so provide additional insights. Jockusch and Simpson provide an extensive array of definitions for one especially interesting degree.

THEOREM 3.40. (Jockusch and Simpson [1976]) *The degree of Kleene's $\mathcal{O}$ (the complete Π^1_1 set) is the largest degree which is below $(\mathbf{a} \vee \mathbf{b})^{(3)}$ whenever $\mathbf{a}, \mathbf{b}$ is an exact pair for $\mathcal{H}$. It is also the largest degree which is below $\mathbf{a}^{(3)}$ for every minimal upper bound $\mathbf{a}$ for $\mathcal{H}$; the smallest degree of the form $(\mathbf{a} \vee \mathbf{b})^{(3)}$ for some exact pair $\mathbf{a}, \mathbf{b}$ for $\mathcal{H}$ or of the form $\mathbf{a}^{(3)}$ for a minimal upper bound $\mathbf{a}$ for $\mathcal{H}$. By relativization, the hyperjump is naturally definable in $\mathcal{D}$ with the jump operator.*

The proofs here use, for example, the nontrivial fact that $X \leq_T \mathcal{O}$ if and only if X is Δ^1_2 over $\mathcal{H}$. Continuing the connection with subsystems of second order arithmetic Jockusch and Simpson also define the ideals I such that M_I is a β-model of Δ^1_n-CA for $n \geq 2$ or of full comprehension by exploiting the fact that these are the ones that are closed under hyperjump. We note that an approach along the lines of controlling initial segments allows us to drop the comprehension assumption.

DEFINITION 3.41. *A nonempty set M of functions from $\mathbb{N}$ to $\mathbb{N}$ is a β-*model *(*of arithmetic*) if it is absolute for Σ^1_1 formulas, i.e. if $S \subseteq (\mathbb{N}^{\mathbb{N}})^3$ is Σ^1_1, $f_1, f_2 \in M$ and $(\exists f)S(f_1, f_2, f)$ then there is an $f \in M$ such that $S(f_1, f_2, f)$.*

THEOREM 3.42. *The property that I is a jump ideal and M_I is a β-model is definable in $\mathcal{D}$ with the jump operator. Indeed, for I a jump ideal M_I is a β-model if and only if for every linearly ordered initial segment J of $\mathcal{D}$ with top in I, if there is an exact pair defining an initial segment K of J with no least element of J above K then there is such an exact pair in I.*

PROOF. First suppose that M_I is a β-model and J is an initial segment of $\mathcal{D}$ with top $\mathbf{a} \in I$. The assertion that there is an exact pair $\mathbf{x}, \mathbf{y}$ for some initial segment K of J with no least element of J above it is Σ^1_1. Thus, if true, there are witnesses for $\mathbf{x}$ and $\mathbf{y}$ in I as I is a β-model. For the other direction suppose S is Σ^1_1, $f_1, f_2 \in M_I$ and $S(f_1, f_2, f)$ holds for some arbitrary f. Now consider the Kleene-Brouwer ordering $KB(S)$ of sequence numbers associated with the Σ^1_1 in $f_1 \oplus f_2$ predicate $S(f_1, f_2, f)$ of f. The ordering is recursive in $f_1 \oplus f_2$ and so (as I is a jump ideal) there is an initial segment J of I with top $\mathbf{a} \in I$ which is isomorphic to $KB(S)$ by an isomorphism in M_I. As there is an f satisfying this predicate, the ordering is not well ordered. Thus there is an initial segment K of J with no least element above it. By Theorem 3.24, there is an exact pair for K. By our assumption there is an exact pair for K in I. Thus there is a descending chain in J arithmetic in this exact pair and $\mathbf{a}$ and so one in M_I. Finally, arithmetically in any such descending chain we can find a g such that $S(f_1, f_2, g)$ and so there is one in M_I as required. (The information needed here about Σ^1_1 predicates and the Kleene-Brouwer ordering is classical but can be found for example in Simpson [1999, Ch. V]. Information about β-models in general can be found in Simpson [1999, Ch. VII].) □

Jockusch and Simpson carry their analysis into the ramified analytic hierarchy which is defined like the constructible hierarchy L in set theory but only involves partial functions on ω as elements and has its definitions restricted to ones analytic over the previously defined level (with parameters allowed although they are not actually necessary). The Δ^1_n master codes H (for an M_I) are the (Turing) complete Δ^1_n over M_I sets, i.e. $X \leq_T H$ iff X is Δ^1_n over M_I. Thus, for example, if $a \in \mathcal{O}$ is a notation for a limit ordinal then H_a is a Δ^1_1 master code for $\{f|(\exists b <_{\mathcal{O}} a)(f \leq_T H_b)\}$ and $\mathcal{O}$ is a Δ^1_2 master code for $\mathcal{H}$.

THEOREM 3.43. (Jockusch and Simpson [1976]) *One can prove analogs of the first definition of $\mathcal{O}$ for the Δ^1_n master codes ($n \geq 2$) as the largest degrees below $(\mathbf{a} \vee \mathbf{b})^{(n+1)}$ for $\mathbf{a}, \mathbf{b}$ exact pairs for the appropriate ideals carried out through the ramified analytic hierarchy. For example, $\mathcal{O}_\omega$, the recursive join of the n^{th} hyperjumps, $\mathcal{O}_n$, of $\mathbf{0}$, is the largest degree $\mathbf{d}$ such that $\mathbf{d} \leq (\mathbf{a} \vee \mathbf{b})^{(3)}$ for every exact pair $\mathbf{a}, \mathbf{b}$ for the ideal generated by the $\mathcal{O}_n$. If λ is the smallest admissible limit of admissibles then $\mathcal{O}_\lambda$, the degree of the complete $\Sigma_1(L_\lambda)$ set, is the largest degree $\mathbf{d}$ such that $\mathbf{d} \leq (\mathbf{a} \vee \mathbf{b})^{(4)}$ for every exact pair $\mathbf{a}, \mathbf{b}$ for the ideal generated by the $\mathcal{O}_\alpha$ for $\alpha < \lambda$.*

The extent to which these results provide natural definitions in $\mathcal{D}$ deserves some further investigation. Jockusch and Simpson proceed through the ramified analytic hierarchy by assuming at limit levels, analogously to what is done in the definition of L, that the ideal I_λ consisting of the union of all the previous levels is given as a predicate of the language and then working with this ideal as an added predicate. It is an interesting question as to when these recursions can be naturally defined in $\mathcal{D}$ or when the ideals themselves have natural independent definitions in $\mathcal{D}$.

Continuing in this vein, Simpson [1977] mentions the following definability result which is remarkable for the depth of the information about constructibility that is needed to establish the definition.

THEOREM 3.44. (Simpson) *There is a natural definition of the relation "$\mathbf{a}$ is constructible from $\mathbf{b}$" in $\mathcal{D}$ with the jump.*

PROOF. (Sketch) Let $\mathcal{C}^{\mathbf{a}}_1$ be the maximal Π^1_1 in $\mathbf{a}$ set with no perfect subset. $\mathbf{b}$ is constructible from $\mathbf{a}$ iff $\mathbf{b}$ is hyperarithmetic in some element $\mathbf{c}$ of $\mathcal{C}^{\mathbf{a}}_1$. Moreover, $\mathbf{c} \in \mathcal{C}^{\mathbf{a}}_1$ iff $\forall \mathbf{b}(\omega_1^{\mathbf{b} \vee \mathbf{a}} = \omega_1^{\mathbf{c} \vee \mathbf{a}} \rightarrow \mathbf{a} \leq_h \mathbf{b})$. We already know how to say that $\mathbf{a} \leq_h \mathbf{b}$ and so only need to define the relation $\omega_1^{\mathbf{x}} = \omega_1^{\mathbf{y}}$. Now for arbitrary $\mathbf{x}$ and $\mathbf{y}$, $\omega_1^{\mathbf{x}} = \omega_1^{\mathbf{y}} \Leftrightarrow \exists \mathbf{z}(\omega_1^{\mathbf{x}} = \omega_1^{\mathbf{x} \vee \mathbf{z}} = \omega_1^{\mathbf{z}} = \omega_1^{\mathbf{y} \vee \mathbf{z}} = \omega_1^{\mathbf{z}})$. Thus it suffices to define this relation if $\mathbf{x} \leq_h \mathbf{y}$. In this case, it is clear that $\omega_1^{\mathbf{x}} = \omega_1^{\mathbf{y}}$ iff $\mathcal{O}^{\mathbf{x}} \nleq_h \mathbf{y}$ and we also already know that $\mathcal{O}^{\mathbf{x}}$, the hyperjump of $\mathbf{x}$, is also definable in $\mathcal{D}$ with the jump operator. □

This result provides a natural statement about $\mathcal{D}$ (with jump) that is independent of ZFC: There is a nonconstructible subset of $\mathbb{N}$. By Corollary 3.38 or

Proposition 3.39, another such statement provided by Simpson is "there is a cone of minimal covers in the hyperdegrees" which is true assuming $0^{\#}$ exists and false in set-forcing extensions of L by Simpson [1975]. These examples prompt a more general question and a specific one.

QUESTION 3.45. (Simpson) *What other natural statements about $\mathcal{D}$ are independent of ZFC?*

QUESTION 3.46. (Simpson) *Is there is a natural definition of the sharp operator in $\mathcal{D}$?*

Along these lines we also mention the analog of Question 2.16. It was also a long standing question first raised as Rogers' [1967] homogeneity conjecture if $\mathcal{D}(\geq \mathbf{a})$, the degrees greater than or equal to $\mathbf{a}$, are isomorphic (or even elementary equivalent, to $\mathcal{D}(\geq \mathbf{b})$ for every $\mathbf{b}$. A negative answer for isomorphisms based on how complicated the base of a cone of minimal covers must be was provided by Shore [1979] and for elementary equivalence by a translation of this issue via codings of models of second order arithmetic in Shore [1982a]. However, we still have no natural difference between any two cones.

QUESTION 3.47. (Simpson) *Is there a natural sentence ϕ of degree theory that is true in some $\mathcal{D}(\geq \mathbf{a})$ but not in every $\mathcal{D}(\geq \mathbf{b})$? If so, for which $\mathbf{a}$ and $\mathbf{b}$ can one find such sentences?*

We close this section by asking for natural definitions of other degrees and ideals that correspond to various types of models of arithmetic.

QUESTION 3.48. *Are the degrees of complete extensions or models of Peano arithmetic or the models of true arithmetic naturally definable in $\mathcal{D}$? Is a countable ideal I of $\mathcal{D}$ being a Scott set, i.e. a model of WKL_0, naturally definable in $\mathcal{D}$?*

4. Parameters and additional predicates

We conclude with a few remarks and questions on (natural) definability in extensions of $\mathcal{R}$ and $\mathcal{D}$. One might first consider definability from parameters. This certainly provides an interesting and important line of investigation. Indeed, in the case of $\mathcal{D}$ the situation (omitting considerations of naturalness) has been fully analyzed by Slaman and Woodin (see Slaman [1991]). Every relation definable in second order arithmetic with parameters is definable in $\mathcal{D}$ with parameters. However, such results are not likely to produce natural definitions in $\mathcal{R}$ or $\mathcal{D}$ itself unless the parameters can later be eliminated (by quantification perhaps) or be defined themselves (as was the case for $\mathbf{0}'$). On the other hand, there are interesting results defining finite sets of degrees in a specified interval of $\mathcal{R}$ such as in Stob [1983] where r.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ are constructed such that $\mathbf{a}_0$ is the unique complement of $\mathbf{a}_1$ in the interval $[\mathbf{0}, \mathbf{a}_0 \vee \mathbf{a}_1]$. This suggests the following question:

QUESTION 4.1. (Li) *Are there definable properties in $\mathcal{R}$ which determine a single degree in some nontrivial interval.*

Another interesting line of investigation is that of adding (a priori) external relations on $\mathcal{R}$ or $\mathcal{D}$ and determining what can then be defined. This is perhaps a more promising line in that the additional relations (like the jump) may eventually be defined or we might be willing to view them as natural themselves in some context. One such class of relations frequently brought into $\mathcal{R}$ is the following.

DEFINITION 4.2. $\mathbf{a}$ *is* n-jump equivalent *to* $\mathbf{b}$*,* $\mathbf{a}\sim_n\mathbf{b}$*, iff* $\mathbf{a}^{(n)} = \mathbf{b}^{(n)}$.

PROPOSITION 4.3. (Nies, Shore and Slaman [1998]) $\mathbf{H}_1$ *is naturally definable from* $\sim_2$ *in* $\mathcal{R}$*:* $\mathbf{a} \in \mathbf{H}_1 \Leftrightarrow (\forall \mathbf{y})(\exists \mathbf{z} \leq \mathbf{a})(\mathbf{z} \sim_2 \mathbf{a})$.

QUESTION 4.4. *Are there any other interesting natural definitions in* $\mathcal{R}$ *that use the relations* $\sim_n$*?*

QUESTION 4.5. *Are any of the relations* $\sim_n$ *themselves naturally definable in* $\mathcal{R}$*? In particular what about* $\sim_2$*?*

An example of an interesting but apparently external relation on $\mathcal{D}$ that has been used for relative definability results is that of a degree of a complete extension of Peano arithmetic. More generally notions and results can be phrased in terms of the relation $\mathbf{a} \ll \mathbf{b}$, i.e. every nonempty $\Pi_1^{0,A}$ class has a member recursive in $\mathbf{b}$. (Note that, by Simpson [1977a, p. 649]), $\mathbf{0} \ll \mathbf{b}$ if and only if $\mathbf{b}$ is the degree of a complete extension of PA if and only if $\mathbf{b}$ is the degree of a set which separates an effectively inseparable pair of r.e. sets.) Here we have an result defining $\mathbf{0}'$ from this relation.

THEOREM 4.6. (Kucera [1988]) $\mathbf{0}' = \inf\{\mathbf{a} \vee \mathbf{b} | \mathbf{0} \ll \mathbf{a}, \mathbf{b} \;\&\; (\mathbf{a} \wedge \mathbf{b}) = \mathbf{0}\}) = \inf\{\mathbf{a} | \mathbf{0} \ll \mathbf{a} \,\&\, \forall \mathbf{c}(\mathbf{0} < \mathbf{c} \leq \mathbf{a} \rightarrow \exists \mathbf{b}(\mathbf{0} \ll \mathbf{b} \leq \mathbf{a} \,\&\, \mathbf{c} \not\leq \mathbf{b}))$.

QUESTION 4.7. *Is the relation* $\mathbf{a} \ll \mathbf{b}$ *(naturally) definable in* $\mathcal{D}$*?*

QUESTION 4.8. *What other interesting natural predicates on* $\mathcal{D}$ *or* $\mathcal{R}$ *might be profitably added to the language?*

References

Ambos-Spies, K. and Fejer, P. [2000], Embeddings of N_5 and the contiguous degrees, to appear.

Ambos-Spies, K. Jockusch, C. G. Jr., Shore, R. A. and Soare, R. I. [1984], An algebraic decomposition of the recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees, *Trans. Amer. Math. Soc.* **281**, 109-128.

Cholak, P. Downey, R. G. and Walk, S. M. [2000] Maximal Contiguous Degrees, to appear.

Cooper, S. B. [1990], The jump is definable in the structure of the degrees of unsolvability (research announcement), *Bull. Am. Math. Soc.* **23**, 151-158.

Cooper, S. B. [1993], On a conjecture of Kleene and Post, *University of Leeds, Department of Pure Mathematics Preprint Series* No. 7.

Cooper, S. B. [1994], Rigidity and definability in the noncomputable universe, in *Logic, Meth. and Phil. of Science IX*, Proc. Ninth Int. Cong. Logic, Meth. and Phil. of Science, Uppsala, Sweden August 7-14, 1991, D. Prawitz, B. Skyrms and D. Westerstahl eds., North-Holland, Amsterdam, 209-236.

Cooper, S. B. [1999], The degrees below $\mathbf{0}'$, in *Handbook of Computability Theory*, E. Griffor ed., North-Holland, Amsterdam, 1999.

Downey, R. G., Jockusch, C. G. Jr. and Stob, M. [1990], Array nonrecursive sets and multiple permitting arguments, in *Recursion Theory Week*, K. Ambos-Spies, G. H. Müller and G. E. Sacks eds., Springer-Verlag, Berlin, 141-174.

Downey, R. G. and Lempp, S. [1997], Contiguity and distributivity in the enumerable Turing degrees, *J. Symb. Logic* **62**, 1215-1240.

Enderton, H. B. and Putnam, H. [1970], A note on the hyperartihmetical hierarchy, *J. Symb. Logic* **35**, 429-430.

Friedberg, R. M. [1957], A criterion for completeness of degrees of unsolvability, *J. Symb. Logic* **22**, 159-60.

Hodes, H. [1982], Uniform upper bounds on ideals of Turing degrees, *J. Symb. Logic* **43**, 601-612.

Hodges, W. [1993], *Model Theory*, Cambridge University Press, Cambridge, England.

Jockusch, C. G. Jr. [1972], Degrees in which the recursive sets are uniformly recursive, *Can. J. Math.* **24**, 1092-1099.

Jockusch, C. G. Jr. and Shore, R. A. [1984], Pseudo-jump operators II: transfinite iterations, hierarchies and minimal covers, *J. Symb. Logic* **49**, 1205-1236.

Jockusch, C. G. Jr. and Shore, R. A. [1985], REA operators, r.e. degrees and minimal covers, in *Recursion Theory, Proc. Symp. Pure Math.* **42**, A. Nerode and R. A. Shore, eds., American Mathematical Society, Providence, RI, 1985, 3-11.

Jockusch, C. G. Jr. and Simpson, S. G. [1976], A degree theoretic definition of the ramified analytic hierarchy, *Ann. Math. Logic* **10**, 1-32.

Jockusch, C. G. Jr. and Soare, R. I., Minimal covers and arithmetical sets, *Proc. Amer. Math. Soc.* **25**, 856-859.

Kucera, A. [1988], On the role of $\mathbf{0}'$ in recursion theory, in *Logic Colloquium '86*, F. R. Drake and J. K. Truss eds., North-Holland, Amsterdam, 133-141.

Lachlan, A. H. [1971], Solution to a problem of Spector, *Can. J. Math.* **23**, 247-256.

Lachlan, A. H. and Soare, R. I. [1994], Models of arithmetic and upper bounds of arithmetic sets, *J. Symb. Logic* **59**, 977-983.

Lerman, M. [1983], *Degrees of Unsolvability*, Springer-Verlag, Berlin.

Lerman, M. [1985], Upper bounds for the arithmetical degrees, *Ann. Pure and Applied Logic* **29**, 225-254.

MacIntyre, J. M. [1977], Transfinite extensions of Friedberg's completeness criterion, *J. Symb. Logic* **42**, 1-10.

Maass, W. [1982], Recursively enumerable generic sets, *J. Symb. Logic*, **47**, 809-823.

Nerode, A. and Shore, R. A. [1980], Second order logic and first order theories of reducibility orderings in *The Kleene Symposium*, J. Barwise, H. J. Keisler and K. Kunen, eds., North-Holland, Amsterdam, 181-200.

Nerode, A. and Shore, R. A. [1980a], Reducibility orderings: theories, definability and automorphisms, *Ann. Math. Logic* **18**, 61-89.

Nies, A., Shore, R. A. and Slaman, T. A. [1998], Interpretability and definability in the recursively enumerable degrees, *Proc. London Math. Soc.* (3) **77**, 241-291.

Posner, D. B. [1980], A survey of non-RE degrees $\leq 0'$, in *Recursion Theory: its Generalizations and Applications, Proc. Logic Col. '79, Leeds, August 1979*, F. R. Drake and S. S. Wainer eds., Cambridge University Press, Cambridge, England, 110-139.

Posner, D. B. [1981], The upper semilattice of degrees $\leq \mathbf{0}'$ is complemented, *J. Symb. Logic*, **46**, 714-722.

Rogers, H. Jr. [1967], Some problems of definability in recursive function theory, in *Sets, Models, and Recursion Theory, Proceedings of the Summer School in Mathematical Logic and 10th Logic Colloquium, Leicester, August-Sept. 1965*, J. N. Crossley ed., North-Holland, Amsterdam, 183-201.

Sacks, G. E. [1963], On the degrees less than $\mathbf{0}'$, *Ann. of Math. (2)* **77**, 211-231.

Sacks, G. E. [1963a], *Degrees of unsolvability*, Annals of Math. Studies **55**, Princeton Univ. Press, Princeton NJ.

Sacks, G. E. [1971], Forcing with perfect closed sets, in *Axiomatic Set Theory, Proc. Symp. Pure Math.* **13**, part I, D. S. Scott ed., AMS, Providence RI, 331-355.

Shore, R. A. [1979], The homogeneity conjecture, *Proc. Nat. Ac. Sci.* **76**, 4218-4219.

Shore, R. A. [1982], Finitely generated codings and the degrees r.e. in a degree $\mathbf{d}$, *Proc. Am. Math. Soc.* **84**, 256-263.

Shore, R. A. [1982a], On homogeneity and definability in the first order theory of the Turing degrees, *J. Symb. Logic* **47,** 8-16.

Shore, R. A. and Slaman, T. A. [2000], Defining the Turing jump, to appear.

Shore, R. A. and Slaman, T. A. [2000a], A splitting theorem for n-REA degrees, to appear.

Simpson, S. G. [1975], Minimal Covers and Hyperdegrees, *Trans. AMS* **209**, 45-64.

Simpson, S. G. [1977], First order theory of the degrees of recursive unsolvability, *Ann. Math. (2)*, **105**, 121-139.

Simpson, S. G. [1977a], Degrees of unsolvability: A survey of results, in *Handbook of Mathematical Logic*, J. Barwise ed., North-Holland, Amsterdam, 631-652.

Simpson, S. G. [1999], *Subsystems of Second Order Arithmetic*, Springer-Verlag, Berlin.

Slaman, T. A. [1991], Degree structures, in *Proc. Int. Cong. Math., Kyoto 1990*, Springer-Verlag, Tokyo, 303-316.

Slaman, T. A. and Steel, J. R. [1989], Complementation in the Turing degrees, *J. Symb. Logic* **54**, 160-176.

Slaman, T. A. and Woodin, H. W. [1986], Definability in the Turing degrees, *Illinois J. Math.* **30**, 320-334.

Slaman, T. A. and Woodin, H. W. [2000], Defining the double jump, to appear.

Soare, R. I. [1987], *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin.

Soare, R. I. [1999], The lattice of recursively enumerable sets, in *Handbook of Computability Theory*, E. Griffor ed., North-Holland, Amsterdam, 1999.

Spector, C. On degrees of recursive unsolvability, *Ann. Math. (2)* **64**, 581-592.

Stob, M. [1983], WTT-degrees and T-degrees of r.e. sets, *J. Symb. Logic* **48**, 921-930.

Department of Mathematics, Cornell Univeristy, Ithaca NY 14853
E-mail address: `shore@math.cornell.edu`
URL: `http://math.cornell.edu/~shore/`