Σ^1_1 in every real in a Σ^1_1 class of reals is Σ^1_1

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Abstract

We first prove a theorem about reals (subsets of $\mathbb{N}$) and classes of reals: If a real X is Σ^1_1 in every member G of a nonempty Σ^1_1 class $\mathcal{K}$ of reals then X is itself Σ^1_1. We also explore the relationship between this theorem, various basis results in hyperarithmetic theory and omitting types theorems in ω-logic. We then prove the analog of our first theorem for classes of reals: If a class $\mathcal{A}$ of reals is Σ^1_1 in every member of a nonempty Σ^1_1 class $\mathcal{B}$ of reals then $\mathcal{A}$ is itself Σ^1_1.

1 Introduction

We work in Cantor space $2^{\mathbb{N}}$ and call its members $X \subseteq \mathbb{N}$, *reals*. We think of members of Baire space $\mathbb{N}^{\mathbb{N}}$ as functions $F : \mathbb{N} \to \mathbb{N}$ (coded as real consisting of pairs of numbers). We use the standard normal form theorems for reals and classes of reals as follows: A real X is Σ^1_1 (in a real G) if it is of the form $\{n | \exists F \forall x R(F \upharpoonright x, x, n)\}$ for a recursive (in G) predicate R. A class $\mathcal{K}$ of reals is Σ^1_1 (in G) if it is of the form $\{X | \exists F \forall x R(X \upharpoonright x, F \upharpoonright x, x)\}$

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for a recursive (in G) predicate R. A real or class of reals is Δ^1_1 (or hyperarithmetic) (in G) if it and its complement are Σ^1_1 (in G). Our first main theorem is the following:

Theorem 2.1. *If a real X is Σ^1_1 in every member G of a nonempty Σ^1_1 class K of reals then X is itself Σ^1_1.*

While the statement of this theorem and certainly the proof we provide in the next section seem to have little to do with either results of hyperarithmetic theory or model theory they are all, in fact, connected along a couple of paths. Indeed, we were thinking about related matters when we proved the theorem.

A basis theorem in recursion theory typically says that every nonempty class of some sort contains a member with some property. For example, the classes may be arbitrary Σ^1_1 classes $\mathcal{K}$ of reals. One, the Gandy Basis Theorem (see Sacks [1990, III.1.5]), says that every nonempty Σ^1_1 class of reals contains one Z such that $\omega_1^Z = \omega_1^{CK}$. (For any Z, ω_1^Z is the least ordinal not recursive, or equivalently not Δ^1_1 in Z; ω_1^{CK} is ω_1^Z for Z recursive (or Δ^1_1).) Another, the Kreisel Basis Theorem (see Sacks [1990, III. 7.2]) , says that if a real X is not hyperarithmetic (i.e. Δ^1_1) then $\mathcal{K}$ also contains a real Z in which X is not Δ^1_1. An equivalent version is that if X is Δ^1_1 in every member of $\mathcal{K}$ then X is Δ^1_1.

Our first theorem is the generalization of the Kreisel basis theorem where Δ^1_1 is replaced by Σ^1_1. (To see that it implies the result of Kreisel note that it says that if X and $\bar{X}$ are both Σ^1_1 (i.e. Δ^1_1) in every member of $\mathcal{K}$ then they are both Σ^1_1 (and so Δ^1_1)). Our theorem also implies the basis result of Gandy: As Kleene's O is not Σ^1_1 there is a $Z \in \mathcal{K}$ in which O is not Σ^1_1. By classical results of Spector (see Sacks [1990, II. 7.7]), this implies that $\omega_1^Z = \omega_1^{CK}$. (See Theorem 2.9.)

Sacks also provides results of hyperarithmetic theory as corollaries to Kreisel's theorem and others that, as he points out, can be viewed as omitting types theorems in ω-logic. They are also immediate consequences of our theorem as we indicate in the next section. We discus these and other related results in next section after we prove our theorem.

After his proof, Sacks [1990, p. 75] says of this connection that "The recursion theorist winding his way through a Σ^1_1 set is a brother to the model theorist threading his way through a Henkin tree." Our proof, which requires no knowledge of either hyperarithmetic theory or model theory, shows that there is another sibling traipsing (or perhaps treading carefully) through a forcing construction.

Our theorem should have been a classical one of hyperarithmetic theory. It also has analogs, both recent and classical, in other settings. When we told Stephen Simpson the result he remarked that Andrews and Miller [2015, Proposition 3.6] had recently proven the analogous result for Π^0_1 classes in place of Σ^1_1 classes. We rephrase it in our terminology as follows:

Theorem 1.1 (Andrews and Miller). *Let P be a nonempty Π^0_1 class. If X is Π^0_1 in every member of P then X is Π^0_1. (Or, equivalently, if X is Σ^0_1 in every member of P*

then X is Σ^0_1.)

Their proof is a forcing proof similar to ours but using Π^0_1 classes instead of Σ^1_1 ones.

At the level of Σ^1_2 classes, a standard basis theorem gives the analogous result (as pointed out to us by John Steel). The classical result (see Moschovakis [1980, 4E.5]) is that the Δ^1_2 reals are a basis for the Σ^1_2 classes of reals. Thus if $\mathcal{K}$ is Σ^1_2 it contains a Δ^1_2 real G and, of course, any real X which is Σ^1_2 in G via Θ is itself Σ^1_2. ($X = \{n|\exists G(\Psi(G)\ \&\ \Theta(G,n))\}$ where Ψ is the Σ^1_2 formula saying G satisfies its Δ^1_2 definition.) Similar basis results hold at higher levels of the projective hierarchy assuming various set theoretic axioms. (See Moschovakis [1980, 5A.4 and 6C.6].)

About the only facts about Σ^1_1 reals and classes that we use in our proof are the the standard normal form theorems mentioned at the beginning of this Introduction.

Our second main theorem is one analogous to Theorem 2.1 but at the level of classes of real.

Theorem 3.1. *If a class $\mathcal{A}$ of reals is Σ^1_1 in every member of a nonempty Σ^1_1 class $\mathcal{B}$ of reals then it is Σ^1_1.*

Our proof of this theorem requires some familiarity with effective descriptive set theory. We give some of the basic facts needed and the proof in §3.

2 The Proof for Reals

We now give the promised forcing style proof of our main theorem.

Theorem 2.1. *If a real X is Σ^1_1 in every member G of a nonempty Σ^1_1 class $\mathcal{K}$ of reals then X is itself Σ^1_1.*

Proof. We use the language of Gandy-Harrington forcing. Forcing conditions are nonempty Σ^1_1 classes $\mathcal{L}$ of reals with set containment as extension. We view the Σ^1_1 formulas $\varphi(G,n)$ as of the form $\exists F\forall x R(G\restriction x, F\restriction x, x, n)$ with R recursive. We say that $\mathcal{L}\Vdash\varphi(G,n)$ if $(\forall Z\in\mathcal{L})(\varphi(Z,n))$. If, as usual, we say $\mathcal{L}\Vdash\neg\varphi(G,n)$ if $(\forall\hat{\mathcal{L}}\subseteq\mathcal{L})(\hat{\mathcal{L}}\nVdash\varphi(G,n))$, this definition is then equivalent to $(\forall Z\in\mathcal{L})(\neg\varphi(Z,n))$. The point here is that if there is a $Z\in\mathcal{L}$ such that $\varphi(Z,n)$ then $\hat{\mathcal{L}}=\mathcal{L}\cap\{Z|\varphi(Z,n)\}$ is a nonempty extension of $\mathcal{L}$ which obviously forces $\varphi(G,n)$.

We now list all the Σ^1_1 formulas $\Theta_k(G,n)$. These are the formulas that could potentially define the reals Σ^1_1 in any G. We consider an X which is a candidate for being Σ^1_1 in every $G\in\mathcal{K}$. We build a sequence $\mathcal{L}_k$ of conditions beginning with $\mathcal{L}_0 = \mathcal{K} = \{G|\exists F_0\forall x R_{m_0}(G\restriction x, F_0\restriction x, x)\}$ as well as initial segments γ_k (of length at least k) of our intended G and $\delta_{i,k}$ of witnesses F_i (of length at least k) showing that $G\in\mathcal{L}_k$. More precisely, each $\mathcal{L}_k$ will be of the form $G\supset\gamma_k\ \&\ \forall i\le k\exists F_i\supset\delta_{i,k}\forall x R_{m_i}(G\restriction x, F_i\restriction x, x)$

for some recursive R_{m_i} (independent of k). Thus, if we successfully continue our construction keeping $\mathcal{L}_k$ nonempty for each k then the $F_i = \lim_k \delta_{i,k}$ for $i \leq k$ will witness that $G = \lim_k \gamma_k$ is in every $\mathcal{L}_k$ as we guarantee that $R_{m_i}(\gamma_k \restriction x, \delta_{i,k} \restriction x, x)$ holds for every $i, x < k$ and every k.

We begin with $\gamma_0 = \emptyset = \delta_{0,0}$ and R_{m_0} as specified by $\mathcal{K}$. So our G will at least be in $\mathcal{K}$ as desired. Suppose we have defined γ_j and $\delta_{i,j}$ for $j, i \leq k$ and wish to define $\mathcal{L}_{k+1}$, γ_{k+1} and $\delta_{i,k+1}$ for $i \leq k+1$ so as to prevent X from being Σ^1_1 in G via Θ_k. We ask if there is an $m \in \omega$ and a nonempty $\mathcal{L} \subseteq \mathcal{L}_k$ such that

1. $m \notin X$ and $\mathcal{L} \Vdash \Theta_k(G, m)$ or
2. $m \in X$ and $\mathcal{L} \Vdash \neg\Theta_k(G, m)$.

Suppose there is such an $\mathcal{L}$ of the form $\exists F_{k+1} \forall x R_{m_{k+1}}(G \restriction x, F_{k+1} \restriction x, x)$. As $\mathcal{L} \subseteq \mathcal{L}_k$ is nonempty we can choose $\gamma_{k+1} \supset \gamma_k$ and $\delta_{i.k+1} \supset \delta_{i,k}$ for $i \leq k$ and some $\delta_{k+1,k+1}$ all of length at least $k+1$ such that $\mathcal{L}_{k+1}$ as given by $G \supset \gamma_{k+1}$ & $(\forall i \leq k+1)(\exists F_i \supset \delta_{i,k+1})(\forall x R_{m_i})(G \restriction x, F_i \restriction x, x)$ is a nonempty subclass of $\mathcal{L}$ (and so, in particular, $R_{m_i}(\gamma_{k+1} \restriction x, \delta_{i,k+1} \restriction x, x)$ for every $i, x \leq k+1$). We can now continue our induction.

Note that if we can successfully define nonempty $\mathcal{L}_k$ in this way for every k then we build a $G = \lim_k \gamma_k$ and $F_i = \lim_k \delta_{i,k}$ for each i such that $\forall x R_{m_i}(G \restriction x, F_i \restriction x, x)$. In particular $\forall x R_{m_0}(G \restriction x, F_0 \restriction x, x)$ and so $G \in \mathcal{K}$. Similarly, $G \in \mathcal{L}_k$ for every $k > 0$. If X is $\Sigma^1_1(G)$ as assumed, then $X = \{n | \Theta_k(G, n)\}$ for some k. We consider the construction at stage $k+1$ and the $\mathcal{L}$ chosen at that stage. If we were in case (1) then as $\mathcal{L} \Vdash \Theta_k(G, m)$ and $G \in \mathcal{L}_{k+1}$, $\Theta(G, m)$ is true but $m \notin X$ for a contradiction. Similarly, if we were in case (2), as $\mathcal{L} \Vdash \neg\Theta_k(G, m)$ and $G \in \mathcal{L}_{k+1}$, $\neg\Theta(G, m)$ is true but $m \in X$ again for a contraction.

Thus we can assume that there is some first stage $k+1$ at which there are no m and $\mathcal{L} \subseteq \mathcal{L}_k$ as required in the construction. In this case we claim that X is Σ^1_1 as desired. Indeed, we claim that X is defined as a Σ^1_1 real by $m \in X \Leftrightarrow (\exists Z \in \mathcal{L}_k)\Theta_k(Z, m)$. To see this suppose first that $(\exists Z \in \mathcal{L}_k)\Theta_k(Z, m)$. Then $\mathcal{L}$ as defined by $\mathcal{L}_k$ & $\Theta_k(G, m)$ is a nonempty Σ^1_1 class such that $\mathcal{L} \Vdash \Theta_k(G, m)$ and so we would have $m \in X$ as desired by the assumed failure of (1) at stage $k+1$ of the construction. On the other hand, if $(\forall Z \in \mathcal{L}_k)(\neg\Theta_k(Z, m)$ then $\mathcal{L}_k \Vdash \neg\Theta_k(G, m)$ and so by the failure of (2) at stage $k+1$ of the construction, $m \notin X$ as desired. $\square$

As usual, we may relativize the Theorem to any real C.

Our theorem easily implies several basic results of hyperarithmetic theory without any appeal to the theory of hyperarithmetic sets as used, for example, in Sacks [1990]. Many of them can also be seen as consequences of type omitting theorems for certain classes of generalized logics. These type omitting arguments are also immediate consequences of our Theorem. We presented two basis theorems of this sort in the introduction and note here that the proof of Kreisel's in Sacks [1990] uses several deep facts about hyperarithmetic reals and Σ^1_1 classes.

Following his proof of the Kreisel basis theorem Sacks [1990] gives as a corollary a result of Kreisel about the intersections of all ω-models of various theories of second order arithmetic from which follow some previous specific results. We state that result now along with some similar ones earlier in Sacks's presentation. These can all be seen as type omitting arguments. After stating them, we explain a general setting which includes them all and give the relevant type omitting theorems as Corollaries of Theorem 2.1.

Theorem 2.2 (Sacks [1990, III. 4.10]). *The intersection of all ω-models of Δ^1_1 comprehension is HYP, the class of all hyperarithmetic sets or equivalently the class of all Δ^1_1 sets.*

More generally, we have the following result of Kreisel.

Theorem 2.3 (Sacks [1990, III.7.3]). *Let K be a Π^1_1 set of axioms in the language of analysis (i.e. second order arithmetic). If a real X belongs to every countable ω-model of K then X is Δ^1_1.*

A similar result is the following.

Theorem 2.4 (Sacks [1990, III.4.13]). *The intersection of all ω-models of Σ^1_1 choice downward closed under many-one reducibility is also HYP.*

In all of these results it is easy to see that the class of models described is Σ^1_1 and, of course, every member X of such a model is recursive in it and so any real in every such model is Σ^1_1 but these models are all trivially closed under complementation. So they all follow from our Theorem.

Moving to the type omitting point of view we, somewhat more generally, consider two sorted logics $(\mathcal{N}, \mathcal{M}, \dots)$ in the usual sense of having two types of variables one ranging over the elements of $\mathcal{N}$ and the other over those of $\mathcal{M}$ in addition to the usual apparatus of function, relation and constant symbols of ordinary first order logic. While formally merely a version of first order logic gotten by adding on predicates for N and M, this logic can be turned into a much stronger one ($\mathcal{N}$-logic) by requiring that all models have their first sort (with some functions and relations on it as given in the structure) isomorphic to some given countable first order structure. The most common example of these logics is ω-logic where we require that $\mathcal{N}$ be isomorphic to the ordinal ω or the standard model $\mathbb{N}$ of arithmetic (depending on the language intended). Again, the most common examples are given by classes of ω-models of fragments T of second order arithmetic as mentioned above. Here, in addition to requiring that $\mathcal{N}$ be the standard model of arithmetic we intend that the elements of $\mathcal{M}$ are subsets of $\mathcal{N}$ and the membership relation $\in$ between members of $\mathcal{N}$ and those of $\mathcal{M}$ is in the language (with the usual axiom of extensionality so that the elements of $\mathcal{M}$ may be identified with true subsets of $\mathcal{N} = \mathbb{N}$). As being an $\mathcal{N}$ model, or even also satisfying some Π^1_1 theory T, is clearly Σ^1_1 in $\mathcal{N}$, we immediately get all the results from Sacks [1990] mentioned above as a corollaries of our theorem. Indeed, we have the following generalization of Kreisel's result in Sacks [1990, III.7.3]:

Theorem 2.5. *If T is a Π^1_1 set of sentences in the two sorted language of $(\mathcal{N}, \mathcal{M}, \ldots)$ and $\mathcal{N}$ is a countable structure for the appropriate sublanguage (restricted to the first sort), T has an $\mathcal{N}$-model and p is a n-type (i.e. a complete consistent set of formulas $\varphi(x)$ with n free variables in the language of $(\mathcal{N}, \mathcal{M}, \ldots)$) which is not Σ^1_1 in $\mathcal{N}$, then there is an $\mathcal{N}$-model of T not realizing p. (Note that, as types are complete sets of formulas, p being Σ^1_1 (in $\mathcal{N}$) is equivalent to its being Δ^1_1 (in $\mathcal{N}$).*

Proof. Being an $\mathcal{N}$ model of T is a Σ^1_1 in $\mathcal{N}$ property and so by our Theorem (relativized to $\mathcal{N}$) there is even an $\mathcal{N}$-model $(\mathcal{N}, \mathcal{M}, \ldots)$ of T in which p is not even Σ^1_1. (Of course, any type realized in $(\mathcal{N}, \mathcal{M}, \ldots)$ is recursive in the complete diagram of $(\mathcal{N}, \mathcal{M}, \ldots)$ and so hyperarithmetic in $(\mathcal{N}, \mathcal{M}, \ldots)$.) $\square$

Viewing our theorem as a type omitting argument suggests that we should be able to omit any countable sequence of types (reals) of the appropriate sort rather than just one. A simple modification of our proof gives the expected result.

Theorem 2.6. *If $\mathcal{K}$ is a nonempty Σ^1_1 class reals and X_n a countable sequence of reals none of which is Σ^1_1, then there is a $G \in \mathcal{K}$ such that no X_n is Σ^1_1 in G. Similarly if no X_n is Δ^1_1, then there is a $G \in \mathcal{K}$ such that no X_n is Δ^1_1 in G.*

Proof. Repeat the proof of the Theorem but at step $k+1 = \langle n, j \rangle$ of the construction replace X by X_n and Θ_k by Θ_j. If we successfully pass through all steps k then the previous argument shows that no X_n is Σ^1_1 in $G \in \mathcal{K}$. On the other hand, if the construction terminates at step $k+1 = \langle n, j \rangle$ then the previous argument shows that X_n is defined as a Σ^1_1 real by $m \in X_n \Leftrightarrow (\exists Z \in \mathcal{L}_k)\Theta_j(Z, m)$ for a contradiction. For the Δ^1_1 version, simply consider the sequence Y_n where $Y_n = X_n$ if X_n is not Σ^1_1 and Y_n is the complement if X_n otherwise (i.e. X_n is not Π^1_1). As now no Y_n is $\Sigma^1_1(G)$, no X_n is $\Delta^1_1(G)$. $\square$

This version of our Theorem also extends the analog of the result actually given by Andrews and Miller [2015, Proposition 3.6].

Of course, we can relativize this theorem as well to any real C. To give a somewhat different example of a such type omitting argument application of this last theorem we provide one for nonstandard models of ZFC for which we have uses elsewhere.

Corollary 2.7. *For every real C and reals X_n not Δ^1_1 in C, there is a countable ω-model of ZFC containing C but not containing any X_n whose well founded part consists of the ordinals less than ω_1^C, the first ordinal not recursive in C.*

Proof. Being a countable ω-model of ZFC containing (a set isomorphic to) C (under the isomorphism taking the ω of the model to true ω) is clearly a Σ^1_1 in C property. Now apply Theorem 2.6 adding on a new real $X_0 = O^C$ (i.e. Kleene's O relativized to C) to the list. It supplies a countable ω-model of ZFC containing C but not containing any of the X_n. As it contains C it contains every ordering recursive in C and so order types for every ordinal less than ω_1^C. On the other hand, if there were an ordinal in the model isomorphic to ω_1^C then, by standard results of hyperarithmetic theory, O^C would be in the model as well.

Finally, we point out that the complexity of the G of Theorem 2.6 (and hence of Corollary 2.7 as well) can be as low as possible. $\square$

Theorem 2.8. *If $\mathcal{K}$ is a nonempty Σ_1^1 class reals and X_n a countable sequence of reals uniformly Δ_1^1 (recursive) in O none of which is Σ_1^1, then there is a $G \in \mathcal{K}$ with G Δ_1^1 (recursive) in O such that no X_n is Σ_1^1 in G. Indeed, G can be chosen to be of strictly smaller hyperdegree than O, i.e. O is not Δ_1^1 in G. As in Theorem 2.6, if we assume only that the X_n are not Δ_1^1 then we may conclude that none are Δ_1^1 in G.*

Proof. Suppose we are at step $k = \langle n, j\rangle$ of the construction. We know that either there is an $m \in X_n$ such that $(\forall Z \in \mathcal{L}_k)(\neg\Theta_k(Z,m))$ or an $m \notin X_n$ such that $(\exists Z \in \mathcal{L}_k)(\Theta_k(Z,m))$. As the X_n are uniformly Δ_1^1 (recursive) in O, and the rest of the conditions considered in the construction are either Σ_1^1 or Π_1^1, O can hyperarithmetically (recursively) decide which case to apply. As choosing the $\gamma_{k+1} \supset \gamma_k$ and $\delta_{i.k+1} \supset \delta_{i,k}$ for $i \leq k$ and so $\mathcal{L}_{k+1}$ now only require finding ones for which the corresponding Σ_1^1 class $\mathcal{L}_{k+1}$ is nonempty, this step is also recursive in O. Of course, as we can add O onto the list of X_n, we then guarantee that O is not Σ_1^1 in G and so, of course, not Δ_1^1 in G as required. $\square$

Note that by a result of Spector's (see Sacks [1990, Theorem II.7.6(ii)]) $\omega_1^{CK} < \omega_1^A$ implies that O is Δ_1^1 in A (indeed there is a pair of Σ_1^1 formulas $\varphi(X,n)$ and $\theta(X,n)$ which define O and its complement for any X with $\omega_1^X > \omega_1^{CK}$), we have the Kleene and Gandy basis theorem for Σ_1^1 classes as well.

Theorem 2.9 (Kleene and Gandy Basis Theorems). *Every nonempty Σ_1^1 class of reals $\mathcal{K}$ contains an element A recursive in and of strictly smaller hyperdegree than O. In particular, one with $\omega_1^A = \omega_1^{CK}$.*

3 The Proof for Classes of Reals

In this section we prove our result for classes of reals.

Theorem 3.1. *If a class $\mathcal{A}$ of reals is Σ_1^1 in every member of a nonempty Σ_1^1 class $\mathcal{B}$ of reals then it is Σ_1^1.*

The proof relies on several basic and important results of effective descriptive set theory. To ease reading the proof, we state the most important ones now. We state lightface versions without parameters. Relativizations to individual real parameters are routine. (Note that, when ordinals or lengths of well-ordered relations are involved, relativization to Z includes replacing ω_1^{CK} by ω_1^Z.) We don't need the full boldface versions. These facts can be found in basic books on effective descriptive set theory such as Moschovakis [1980], higher recursion theory such as Sacks [1990] or Hinman [1978] or even reverse mathematics such as Simpson [2009].

Proposition 3.2 (Codes). *We can code Δ_1^1 classes of reals $\mathcal{V}$ as either Δ_1^1 reals C (Δ_1^1 codes) or as numbers e by coding the Δ_1^1 code C as a number e (hyperarithmetic codes for Δ_1^1 reals). In either case, the property of being a code is Π_1^1 and membership of a real Z in the set coded by C or e is a Δ_1^1 relation given that C and e are codes. Similarly, membership of a number n in a Δ_1^1 real with hyperarithmetic code e is a Δ_1^1 relation. We can pass in a Δ_1^1 way between these types of codes and the syntactic ones given by the formulas required in our definition of Δ_1^1 reals and classes given that all the objects are, in fact, codes. In this situation we often abuse notation by writing $Z \in C$ to denote the assertion that Z is in the class coded by C. When C and D are both codes, we use $D \subseteq C$ to denote the assertion that $\forall Z(\ Z \in D \to Z \in C)$ and similarly for $D \supseteq C$. These relations are then all Π_1^1. These facts also imply that the predicate Z is $\Delta_1^1(X)$ is Π_1^1. (We also use $C \supseteq \mathcal{A}$ for an arbitrary class $\mathcal{A}$ of reals to mean that every real in the set coded by C is in $\mathcal{A}$.)*

Proposition 3.3 (Representation Theorem). *If $\mathcal{V}$ is a Π_1^1 class then there is a Δ_1^1 function $\mathcal{F}$ such that $Z \in \mathcal{V} \Leftrightarrow \mathcal{F}(Z) \in WO$ where WO is the class of reals Z which, viewed as a set of pairs of numbers, represents a well ordering. If $Z \in WO$, we write $|Z|$ for the ordinal represented by Z.*

Proposition 3.4 (Bounding). *If $\mathcal{V}$ is a Π_1^1 class of reals, $\mathcal{F}$ is as in Proposition 3.3 then $\mathcal{V}$ is Δ_1^1 if and only if there is a bound $< \omega_1^{CK}$ on the order types of $\mathcal{F}(Z)$ for $Z \in \mathcal{V}$. Moreover, if $\mathcal{V}$ contains only Δ_1^1 reals and $\mathcal{G}$ is a Δ_1^1 subset of $\mathcal{V}$ then such a bound (expressed as either a real or a number coding a recursive well-ordering) for $\{\mathcal{F}(Z)|Z \in \mathcal{G}\}$ (or even any Σ_1^1 subset of the Δ_1^1 well orderings) can be found in a Δ_1^1 way from the codes (or indices) for $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{V}$. As a consequence we may, in this case, divide $\mathcal{V}$ into an increasing,continuous sequence $\left\langle \mathcal{V}_i | i < \omega_1^{CK} \right\rangle$ of uniformly Δ_1^1 sets given by $\mathcal{V}_i = \{Z \in \mathcal{V}|\ |\mathcal{F}(Z)| < i\}$.*

Remark 3.5. While we have not found an explicit statement in our references of the uniformity described in this bounding theorem, it can easily be deduced from the uniform version of the analogous theorem for sets of numbers (as in e.g. Sacks [1990, II.3.4]) by translating the real codes for ordinals $< \omega_1^{CK}$ to numbers in O of at least as large a rank given by Sacks [1990, I.4.3].

Proposition 3.6 (Gandy-Harrington Forcing). *We can define a general notion of forcing whose conditions are Σ_1^1 classes ordered by inclusion as extension. A simplified version of the proof of Theorem 2.1 that leaves out the diagonalization requirements shows that we may construct a generic G in any given Σ_1^1 class meeting any countable collection* *of dense sets. Thus we may use this forcing notion in any of the common ways. As usual, we will be interested in forcing over countable standard models of fragments of ZFC containing various specified reals. In addition to the typical results about forcing such as forcing equals truth, we note that, by the arguments in the proof of Theorem 2.1, a Π^1_1 sentence $\varphi(G)$ about the generic G is forced by a condition (Σ^1_1 set) $\mathcal{P}$ if and only if $\forall Z \in \mathcal{P}(\varphi(Z))$. We also note that if $\langle G_0, G_1\rangle$ is generic then both G_0 and G_1 are generic. (See Miller [1995, §30] for more about this forcing notion and Lemma 30.3 there for this last particular fact.) Absoluteness considerations will also play a role in our applications of this forcing.*

As a notational convenience in proving our theorem, we can, by the Gandy basis theorem (Theorem 2.9) and the fact that $\omega_1^B = \omega_1^{CK}$ is a Σ^1_1 predicate (of B), assume without loss of generality that $\omega_1^B = \omega_1^{CK}$ for every $B \in \mathcal{B}$. (Note $\omega_1^B = \omega_1^{CK} \Leftrightarrow \forall e(\{e\}^B$ is a well-ordering $\rightarrow \exists i \exists f(f$ is an isomorphism of $\{i\}$ and $\{e\}^B$).)

We begin with some crucial approximations to our class $\mathcal{A}$ and an analysis of their properties.

Notation 3.7. *We let $\mathcal{D}_B = \{C | C$ is a $\Delta^1_1(B)$ code & $C \supseteq \mathcal{A}\}$ and $\mathcal{A}_B = \{A | (\forall C \in \mathcal{D}_B)(A \in C)\}$. Similarly, we let $\mathcal{A}_0 = \{A | A$ is a member of every Δ^1_1 class containing $\mathcal{A}\}$. For $B \in \mathcal{B}$, we let $\psi_B(Z)$ be a $\Sigma^1_1(B)$ formula defining $\mathcal{A}$.*

Lemma 3.8. *For $B \in \mathcal{B}$, $\mathcal{D}_B$ is $\Pi^1_1(B)$ and $\mathcal{A}_B$ is $\Sigma^1_1(B)$.*

Proof. Fix $B \in \mathcal{B}$. For any real C, $C \in \mathcal{D}_B$ if and only if C is a $\Delta^1_1(B)$ code and $\forall Z(\psi_B(Z) \rightarrow Z \in C)$. As ψ_B is $\Sigma^1_1(B)$, both conjuncts here are $\Pi^1_1(B)$ by Proposition 3.2 and so $\mathcal{D}_B$ is $\Pi^1_1(B)$ as required. The second claim now follows directly from the definition of $\mathcal{A}_B$ and Proposition 3.2. (Rephrase the definition of $\mathcal{D}_B$ in terms of number codes to make the quantifier count work.) □

Lemma 3.9. *For any reals A and B in $\mathcal{B}$ with $\omega_1^{A,B} = \omega_1^{CK}$, $A \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}_B$.*

Proof. Clearly $A \in \mathcal{A}$ implies that $A \in \mathcal{A}_B$ for every $B \in \mathcal{B}$ by the definition of $\mathcal{A}_B$. For the other direction suppose $A \notin \mathcal{A}$. Let $\mathcal{F} \in \Delta^1_1(B)$ be as in Proposition 3.3 for the $\Pi^1_1(B)$ class $\mathcal{V} = \{Z | \neg\psi_B(Z)\}$ which is the complement of $\mathcal{A}$. Thus $\mathcal{F}(A)$ is a well ordering which is $\Delta^1_1(A, B)$ and so, by hypothesis, less than some $i < \omega_1^{CK}$. Thus the (obviously Σ^1_1) set $\{Z | \mathcal{F}(Z) < i\}$ is equal to the Π^1_1 set $\{Z | \neg\psi_B(Z)$ & $(i+1) \not\leq \mathcal{F}(Z)\}$ and so is $\Delta^1_1(B)$. It is disjoint from $\mathcal{A}$ and has A as a member by our choice of i. The $\Delta^1_1(B)$ code C for its complement is then a member of $\mathcal{D}_B$ not containing A as a member. This C is then a witness that $A \notin \mathcal{A}_B$ as required. □

Lemma 3.10. *$\mathcal{A}_0$ is Σ^1_1.*

Proof. Consider the real $J = \{e | e$ is a hyperarithmetic index for a Δ^1_1 code of a superset of $\mathcal{A}\} = \{e | e$ is a hyperarithmetic index for a real in $\mathcal{D}_B\}$. This real J is $\Pi^1_1(B)$ in every

$B \in \mathcal{B}$ by Lemma 3.8 and Proposition 3.2. So by Theorem 2.1 (formally applied to the complements) is Π^1_1. Thus $\mathcal{A}_0$ which is the intersection of the sets coded by indices in J is Σ^1_1: $Z \in \mathcal{A}_0 \Leftrightarrow \forall e(e \in J \rightarrow Z$ is in the set coded by e). $\square$

Lemma 3.11. *If $B, C \in \mathcal{B}$ and $\omega_1^{B,C} = \omega_1^{CK}$, then $\mathcal{A}_B = \mathcal{A}_C$.*

Proof. If not, then we have, without loss of generality, an $A \in \mathcal{A}_C - \mathcal{A}_B$. So there is a code $D \in \mathcal{D}_B$ with $A \notin D$ and $A \in \mathcal{A}_C$. Now the nonempty class $\mathcal{W} = \{Z | Z \in \mathcal{A}_C \;\&\; Z \notin D\}$ is $\Sigma^1_1(B, C)$ by Lemma 3.8 and Proposition 3.2. Thus by the Gandy basis theorem (relative to B, C) (Theorem 2.9) there is a W in $\mathcal{W}$ and so in $\mathcal{A}_C - \mathcal{A}_B$ with $\omega_1^{W,B,C} = \omega_1^{CK}$. Lemma 3.9, however, tells us that $W \in \mathcal{A}_B \Leftrightarrow W \in \mathcal{A} \Leftrightarrow W \in \mathcal{A}_C$ for a contradiction. $\square$

Lemma 3.12. *If $B, C \in \mathcal{B}$ and $\omega_1^{B,C} = \omega_1^{CK}$, then for every $X \in \mathcal{D}_B$ there is a $Y \in \mathcal{D}_B$ and a $Z \in \mathcal{D}_C$ such that $Y \subseteq X$ and $Z \subseteq Y \subseteq Z$, i.e. Y and Z are codes for the same set.*

Proof. By Proposition 3.4, there is a uniformly Δ^1_1 continuous increasing sequences $\mathcal{D}_{B,i}$ ($i < \omega_1^{CK} = \omega_1^B$) with union $\mathcal{D}_B$. We can then set $\mathcal{A}_{B,i} = \{A | \forall C \in \mathcal{D}_{B,i}(A \in C)\}$. This sequence is clearly nested and continuous with intersection $\mathcal{A}_B$. As $\mathcal{D}_{B,i}$ and all its members are $\Delta^1_1(B)$, the $\mathcal{A}_{B,i}$ are also uniformly $\Delta^1_1(B)$ by Proposition 3.2 as we can convert to number codes. Similarly, we have $\mathcal{D}_{C,i}$ and $\mathcal{A}_{C,i}$ ($i < \omega_1^{CK} = \omega_1^C$). By Lemma 3.11 we know that for each $i < \omega_1^{CK}$ and $Z \in \overline{\mathcal{A}_{B,i}}$ there is a $j < \omega_1^{CK}$ such that $Z \in \overline{\mathcal{A}_{C,j}}$. By Proposition 3.4, there is a $k < \omega_1^{CK}$ such that for every $Z \in \overline{\mathcal{A}_{B,i}}$, $Z \in \overline{\mathcal{A}_{C,k}}$ and we can get k uniformly $\Delta^1_1(B, C)$. Of course, the analogous fact switching B and C is also true. Iterating and interleaving these $\Delta^1_1(B, C)$ functions starting with any $i < \omega_1^{CK}$ produces a $\Delta^1_1(B, C)$ increasing sequence of $k < \omega_1^{CK}$. By Proposition 3.4, this sequence has a bound and hence a supremum $l < \omega_1^{CK}$ and $\mathcal{A}_{B,l} = \mathcal{A}_{C,l}$.

Now consider any $X \in \mathcal{D}_{B,i}$ so $X \supseteq \mathcal{A}_{B,l}$ for any $l > i$ in ω_1^{CK}. We may now choose one such that $\mathcal{A}_{B,l} = \mathcal{A}_{C,l}$. As $\mathcal{A}_{B,l} \in \Delta^1_1(B)$ and contains $\mathcal{A}$, there is a code $Y \in \mathcal{D}_B$ for it. Similarly there is a code $Z \in \mathcal{D}_C$ for $\mathcal{A}_{C,l}$. As $\mathcal{A}_{B,l} = \mathcal{A}_{C,l}$, these are then the desired Y and Z. $\square$

Lemma 3.13. *For every $B \in \mathcal{B}$, $\mathcal{A}_B = \mathcal{A}_0$.*

Proof. Fix $B \in \mathcal{B}$. Clearly, it suffices to prove that $\forall X \in \mathcal{D}_B \exists Y \in \mathcal{D}_B (X \supseteq Y \;\&\; \mathcal{V} = \{Z | Z \in Y\} \in \Delta^1_1)$. (As this says there is, for each $X \in \mathcal{D}_B$, a Δ^1_1 code V for a Δ^1_1 class $\mathcal{V}$ contained in the class coded by X and containing $\mathcal{A}$ (as $Y \in \mathcal{D}_B$). This code shows that $\mathcal{A}_0 \subseteq \mathcal{A}_B$ by definition. On the other hand, $\mathcal{A}_B \subseteq \mathcal{A}_0$ for every B.)

Fix an $X \in \mathcal{D}_B$. Consider now the class $\mathcal{W} = \{C \in \mathcal{B} | (\forall Y \in \mathcal{D}_B)(X \supseteq Y \rightarrow \{Z | Z \in Y\} \notin \Delta^1_1(C)\}$. By Proposition 3.2, this class is $\Sigma^1_1(B)$. If it were nonempty then, by the Gandy basis theorem (relative to B) (Theorem 2.9), it would have a member C with $\omega_1^{B,C} = \omega_1^{CK}$. This would provide a counterexample to Lemma 3.12 and so $\mathcal{W}$ is empty.

We now work with a countable standard model which contains B and satisfies a fragment of ZFC sufficient to guarantee the absoluteness of Σ^1_1 formulas. Note, for example, that all reals Δ^1_1 in B (and so all in $\mathcal{D}_B$) are in this model.

Let $G \in \mathcal{B}$ be a Gandy-Harrington generic over this model as in Proposition 3.6. As $G \notin \mathcal{W}$, there is a $Y \in \mathcal{D}_B$ such that $X \supseteq Y$ and $\{Z|Z \in Y\} \in \Delta^1_1(G)$. Fix a specific Δ^1_1 definition of this class from G, i.e. $\Sigma^1_1(G)$ formulas φ and θ such that $\forall Z(\varphi(G,Z) \leftrightarrow \neg\theta(G,Z))$, $\forall Z(Z \in Y \rightarrow \varphi(G,Z))$ and $\forall Z(Z \notin Y \rightarrow \theta(G,Z))$. As G is generic we have a Σ^1_1 $\mathcal{P}$ forcing these sentences. Now consider the Σ^1_1 class $\mathcal{Q} = \{\langle C, C'\rangle \,|C, C' \in \mathcal{P} \;\&\; \exists Z(\varphi(C,Z) \;\&\; \theta(C',Z) \;\vee\; \varphi(C',Z) \;\&\; \theta(C,Z))\}$. If $\mathcal{Q}$ is nonempty then there is a Gandy-Harrington generic $\langle C, C'\rangle \in \mathcal{Q}$. Each of C and C' is in $\mathcal{P}$ and Gandy-Harrington generic by Proposition 3.6. Thus any Z witnessing that $\langle C, C'\rangle \in \mathcal{Q}$ would be a counterexample to one of the sentences above forced by $\mathcal{P}$ and hence true of C and C'. Thus $\mathcal{Q}$ is empty and so $Z \in Y \Leftrightarrow (\forall C \in \mathcal{P})(\neg\theta(C,Z))$ and $Z \notin Y \Leftrightarrow (\forall C \in \mathcal{P})(\neg\varphi(C,Z))$ and $\{Z|Z \in Y\}$ is Δ^1_1 as required. □

We now prove our theorem on Σ^1_1 classes.

Proof of Theorem 3.1: We claim that $A \in \mathcal{A}$ if and only if $A \in \mathcal{A}_0$ (which is Σ^1_1 by Lemma 3.10) and one of the following two Σ^1_1 statements hold for a Σ^1_1 formula ψ that we will define below:

(1) $\omega_1^A = \omega_1^{CK}$ or

(2) $\omega_1^A > \omega_1^{CK} \rightarrow \psi(A)$.

Now $A \in \mathcal{A} \rightarrow A \in \mathcal{A}_0$ by the definition of $\mathcal{A}_0$. So we may assume that $A \in \mathcal{A}_0$ and show that $A \in \mathcal{A} \Leftrightarrow$ (1) or (2) holds. If (1) holds then by the Gandy basis theorem (Theorem 2.9) (relative to A) we may choose a $B \in \mathcal{B}$ with $\omega_1^{A,B} = \omega_1^{CK}$. Now by Lemma 3.9, $A \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}_B$ while $A \in \mathcal{A}_B \Leftrightarrow A \in \mathcal{A}_0$ by Lemma 3.13. Thus, in this case, $A \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}_0$ as required.

Assume then that (1) fails and so the hypothesis of (2) holds. We now must argue that we have a Σ^1_1 formula $\psi(A)$ that, under these assumptions, is equivalent to $A \in \mathcal{A}$. As mentioned just before Theorem 2.9, there is a pair if Σ^1_1 formulas $\varphi(X,n)$ and $\theta(X,n)$ which define O and its complement for any X with $\omega_1^X > \omega_1^{CK}$. By the Kleene basis theorem (Theorem 2.9) there is a recursive index computing a $B \in \mathcal{B}$ from O. By the hypothesis of our theorem there is a $\Sigma^1_1(B)$ formula $\psi_B(Z)$ defining $\mathcal{A}$. Thus there is a Σ^1_1 formula $\hat{\psi}(X,Z)$ which defines $\mathcal{A}$ from any X with $\omega_1^X > \omega_1^{CK}$. We now take our desired ψ to be $\hat{\psi}(A,A)$. □

As a final comment, we point out that if we had only wanted to prove Theorem 3.1 in the Δ^1_1 case we would have a simple proof along the lines of the last paragraph of the proof of Lemma 3.13. This argument also gives a proof of the analog for classes of reals of the Δ^1_1 case of Theorem 2.6.

Theorem 3.14. *If $\mathcal{B}$ is a nonempty Σ^1_1 class of reals and $\mathcal{X}_n$ a countable sequence of classes of reals none of which is Δ^1_1, then there is a $G \in \mathcal{B}$ such that no $\mathcal{X}_n$ is Δ^1_1 in G.*

Proof. Note that if $\mathcal{X}_m \notin \Delta^1_1(B)$ for every $B \in \mathcal{B}$ then any $G \in \mathcal{B}$ works for $\mathcal{X}_m$. Thus we may assume that for every n there is a $B_n \in \mathcal{B}$ and φ_n and θ_n Σ^1_1 formulas with two free real variables which, with B_n for the first variable, define $\mathcal{X}_n$ and its complement. Let $G \in \mathcal{B}$ be a Gandy-Harrington generic over a countable standard model of a sufficient fragment of ZFC containing the B_n. We claim no $\mathcal{X}_n$ is $\Delta^1_1(G)$. If not, let φ and θ be $\Sigma^1_1(G)$ formulas defining some $\mathcal{X}_n$ and its complement. Let $\mathcal{P}$ be a condition which forces that $(\forall Z)(\varphi_n(B_n, Z) \rightarrow \varphi(G, Z))$, $(\forall Z)(\theta_n(B_n, Z) \rightarrow \theta(G, Z))$ and $(\forall Z(\varphi(G, Z) \leftrightarrow \neg\theta(G, Z))$. Now consider the Σ^1_1 class $\mathcal{Q} = \{\langle C, C'\rangle \,|C, C' \in \mathcal{P}\ \&\ \exists Z(\varphi(C, Z)\ \&\ \theta(C', Z)\ \vee\ \varphi(C', Z)\ \&\ \theta(C, Z))\}$. If $\mathcal{Q}$ is nonempty then there is a Gandy-Harrington generic $\langle C, C'\rangle \in \mathcal{Q}$. Each of C and C' is in $\mathcal{P}$ and Gandy-Harrington generic by Proposition 3.6. Thus any Z witnessing that $\langle C, C'\rangle \in \mathcal{Q}$ would be a counterexample to one of the sentences above forced by $\mathcal{P}$ and hence true of C and C'. Thus $\mathcal{Q}$ is empty and so $Z \in \mathcal{X}_n \Leftrightarrow (\forall C \in \mathcal{P})(\neg\theta(C, Z))$ and $Z \notin \mathcal{X}_n \Leftrightarrow (\forall C \in \mathcal{P})(\neg\varphi(C, Z))$ and so $\mathcal{X}_n$ is Δ^1_1 for the desired contradiction. □

Corollary 3.15. *Any class $\mathcal{A}$ of reals which is Δ^1_1 in every member of a Σ^1_1 class $\mathcal{B}$ of reals is Δ^1_1.*

We do not know if the full analog of Theorem 2.6 for classes of reals , i.e. Theorem 3.14 with Δ^1_1 replaced by Σ^1_1, is also true.

4 Bibliography

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